

## Math 22, Summer 2013, Final Exam

Name (Print):

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Last

First

**Instructions:** You are not allowed to use calculators, books, or notes of any kind. You may not look at a classmate's exam for "inspiration." You must explain your reasoning behind each solution to receive full credit. Credit will not be awarded for correct answers with no explanation (with the exception of problem #1).

You may use pages 14-17 of the exam as scratch paper, but any work you intend to be graded should be on the exam itself in the space provided. If you run out of room, clearly indicate which page of scratch paper your solution is on and circle the solution that should be graded.

Before beginning the exam, skim through the problems to verify that you have one true/false question and nine free-response questions.

Problem #	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
BONUS	2	
Total	50	

1. Determine whether each statement below is true or false and indicate your answer by circling the appropriate choice (1pt each):
- (a) ( True / False ) Suppose  $A$  is an  $n \times n$  matrix such that  $A\mathbf{x} = \mathbf{b}$  is inconsistent for some  $\mathbf{b} \in \mathbb{R}^n$ . Then  $A\mathbf{x} = \mathbf{0}$  has a nontrivial solution.
  - (b) ( True / False ) If  $A$  is an  $n \times n$  symmetric matrix, then the dimensions of the eigenspaces of  $A$  must sum to  $n$ .
  - (c) ( True / False ) Suppose  $A$  is a  $3 \times 5$  matrix. Then the columns of  $A$  are linearly independent.
  - (d) ( True / False ) The columns of an invertible  $n \times n$  matrix form a basis for  $\mathbb{R}^n$ .
  - (e) ( True / False ) If  $A$  is row equivalent to  $B$ , then  $\text{Col}A = \text{Col}B$ .

2. Consider the set  $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} \right\}$

(a) Is  $S$  linearly independent? Explain.

3. For what value(s) of  $h$  will  $\mathbf{y}$  be in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ , if  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ ,

and  $\mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$ .

4. (a) Determine the standard matrix for the linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that first reflects the points in  $\mathbb{R}^2$  across the  $x_1$ -axis, and then reflects the points across the line  $x_2 = x_1$ . Show your work. (2pts)

(b) Draw a possible echelon form for the standard matrix of a linear transformation  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  that is onto, using '□' to represent pivot positions, '0' for entries that must be zero, and '\*' for the remaining entries. (1pt)

(c) Can  $T$  in part (b) be one-to-one? Why or why not? (2pts)

5. Let  $T : V \rightarrow W$  be a linear transformation from a vector space  $V$  into a vector space  $W$ . Prove that the range of  $T$  is closed under vector addition. [Hint: if  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in the range of  $T$ , this implies that there exist vectors  $\mathbf{x}$  and  $\mathbf{y}$  in the domain  $V$  of  $T$  such that  $\mathbf{u} = T(\mathbf{x})$  and  $\mathbf{v} = T(\mathbf{y})$ ].

6. (a) If  $A$  is a  $5 \times 9$  matrix, what is the smallest possible dimension of  $\text{Nul}A$ ? Explain.

(b) A scientist solves a nonhomogeneous system of 8 linear equations in 10 unknowns and finds that 4 of the unknowns are free variables. Can this scientist be certain that if the right hand side of the equations are changed, the new nonhomogeneous system will have a solution? Explain.

(c) Suppose the matrix  $A$  is row equivalent to  $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ . Determine  $\text{rank}A$ ,  $\text{dimNul}A$ ,  $\text{dimCol}A$ , and  $\text{dimRow}A$ .



7. Consider the set  $\mathcal{B} = \{p_1, p_2, p_3\}$ , where  $p_1(t) = 1 - 3t^2$ ,  $p_2(t) = 2 + t - 5t^2$ , and  $p_3(t) = -1 - 2t + t^2$ . Let  $\mathcal{E} = \{1, t, t^2\}$ , the standard basis for  $\mathbb{P}_2$ . Using the fact that the linear transformation  $[\ ]_{\mathcal{E}}$  is an isomorphism from the vector space  $\mathbb{P}_2$  onto the vector space  $\mathbb{R}^3$ , determine whether  $\mathcal{B}$  is a linearly independent or a linearly dependent set in  $\mathbb{P}_2$ .

8. Let  $\mathbf{y} = \begin{bmatrix} 12 \\ 14 \\ 25 \end{bmatrix}$ ,  $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ , and  $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ . Let  $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ .

(a) Show that  $\{\mathbf{u}_1, \mathbf{u}_2\}$  is an orthogonal set. (1pt)

(b) Write  $\mathbf{y}$  as  $\hat{\mathbf{y}}_1 + \hat{\mathbf{y}}_2$ , where  $\hat{\mathbf{y}}_1 \in W$  and  $\hat{\mathbf{y}}_2 \in W^\perp$ . (4pts)

9. Let  $A = \begin{bmatrix} 4 & 4 \\ 0 & 4 \\ -4 & 4 \end{bmatrix}$ . The eigenvalues of  $A^T A$  are  $\lambda_1 = 48$  and  $\lambda_2 = 32$  (Note:  $48 = 16 \cdot 3$  and  $32 = 16 \cdot 2$ ). Let  $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .  $\{\mathbf{v}_i\}$  is a basis for the eigenspace of  $A^T A$  associated with  $\lambda_i$ ,  $i = 1, 2$ . Suppose  $A = U\Sigma V^T$  is a singular value decomposition of  $A$ .

(a) Determine  $V$ . (1pt)

(b) Determine  $\Sigma$ . (2pt)

(c) Determine  $U$ . (2pt)

10. Let  $\{\mathbf{v}_1, \mathbf{v}_2\}$  be a basis for the subspace  $W$  of  $\mathbb{R}^3$ , where  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ . Use the Gram-Schmidt process to determine an orthogonal basis for  $W$ .

**BONUS** Let  $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ , where  $b \neq 0$ .  $A$  is a symmetric matrix. Show that  $A$  is orthogonally diagonalizable.

[Note: do not use the theorem stating that a matrix is orthogonally diagonalizable iff the matrix is symmetric!]









