Math 22 Fall 2013 Final Exam
Friday, November 22, 2013

PRINT NAME: ________________________________

INSTRUCTIONS: READ CAREFULLY!

This is a closed book, closed notes exam. Use of calculators is not permitted. You have three hours, do all problems.

On all **free response** questions below you must show your step-by-step work and make sure it is clear *how* you arrived at your solution. Whenever you answer a question, don't just say "Yes" or "No", but always justify your answers. No credit is given for solutions without appropriate work or justification. You will receive partial credit for partially correct answers.

For multiple choice and True/False questions, no justification is necessary. Therefore, **leave no multiple choice question unanswered!** Guessing is allowed: A wrong guess does not cost you more points than leaving the question unanswered.

The Honor Principle requires that you neither give nor receive any aid on this exam.

GOOD LUCK!

FERPA waiver: By my signature I relinquish my FERPA rights in the following context: This exam paper may be returned en masse with others in the class and I acknowledge that I understand my score may be visible to others. If I choose not to relinquish my FERPA rights, I understand that I will have to present my student ID at my instructors office to retrieve my examination paper.

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Grader's use only:

1. ________ /11

2. ________ /10

3. ________ /14

4. ________ /4

5. ________ /4

6. ________ /3

7. ________ /3

8. ________ /7

9. ________ /4

10. ________ /7

11. ________ /5

12. ________ /3

13. ________ /7

14. ________ /5

15. ________ /8

16. ________ /10

Total: ________ /105

1. (a) [5 pt] Give a *parametric description* of the solution of the following linear system:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & + & 3x_3 & & & = & 3 \\ -2x_1 & - & x_2 & & & + & 2x_4 & = & 6 \\ 4x_1 & + & 2x_2 & + & 9x_3 & + & 3x_4 & = & 20 \end{array}$$

(b) [3 pt] Does the following system have a solution for all possible values of a, b, c? Explain your answer.

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & + & 3x_3 & & & = & a \\ -2x_1 & - & x_2 & & & + & 2x_4 & = & b \\ 4x_1 & + & 2x_2 & + & 9x_3 & + & 3x_4 & = & c \end{array}$$

(c) [3 pt] If $a = b = c = 0$ in the system above, what is the dimension of the solution set?

2. (a) [4 pt] What is the determinant of the matrix A?

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

(A) 0 (B) 1 (C) 36 (D) 48 (E) 72 (F) 120

ANSWER

(b) [2 pt] Is A invertible?

(c) [2 pt] Is A diagonalizable?

(d) [2 pt] What is Nul$A$?

3. The linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $T(\mathbf{x}) = A\mathbf{x}$ has matrix

$$A = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}$$

(a) [3 pt] The vector $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue?

(b) [4 pt] If C is a solid figure in $\mathbb{R}^3$ with volume 8, what is the volume of the image C' of C under the transformation T?

(c) [2 pt] Is T one-to-one? Is T onto?

(d) [5 pt] Let $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation $S(x_1, x_2, x_3) = (x_1, x_2, 0)$. Find the standard matrix of the composite transformation $\mathbb{R}^3 \to \mathbb{R}^3$, which is obtained if we *first* apply the transformation S and *then* the transformation T.

4. [4 pt] Which of the following statements is *not* logically equivalent to the statement

"The $n \times n$ matrix A is invertible."

(A) The column vectors of A are linearly independent.

(B) Nul$A$ contains only one point.

(C) A^{-1} is invertible.

(D) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.

(E) The rank of A is not zero.

(F) A^T is invertible.

ANSWER

5. [4 pt] Is the vector $\mathbf{v}$ in the subspace Span$\{\mathbf{b_1}, \mathbf{b}_2\}$?

$$\mathbf{b}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$$

6. [3 pt] Suppose A is a 4×5 matrix, and Nul$A$ is a 2 dimensional subspace of $\mathbb{R}^5$. What is the dimension of the subspace of $\mathbb{R}^4$ consisting of all vectors $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ is a consistent system?

7. [3 pt] If $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation $T\colon\ \mathbb{R}^3 \to \mathbb{R}^4$, and Nul$A$ is a *plane* in $\mathbb{R}^3$, then what is the dimension of Col$A$?

8. Let W be the plane in $\mathbb{R}^3$ spanned by the orthogonal vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

(a) [4 pt] Which point on the plane W is closest to point $(1, 0, 1)$?

(b) [3 pt] What is the distance between $(1, 0, 1)$ and W?

9. [4 pt] Suppose that P is an arbitrary orthogonal $n \times n$ matrix and D is an arbitrary diagonal $n \times n$ matrix. Show that $A = PDP^{-1}$ is always a symmetric matrix.

10. [7 pt] The relation between two variables x, y is modeled theoretically by the formula $y = \beta_0 + \beta_2 x^2$. Find the curve $y = \beta_0 + \beta_2 x^2$ that is the best fit (in the sense of least-squares approximations) for the following set of (x, y) data:

$$(-1, 1),\ (0, -2),\ (1, -2),\ (2, 0)$$

11. [5 pt] For which values of x, y is the matrix A diagonalizable?

$$A = \begin{pmatrix} 5 & 2 & x \\ 0 & 1 & y \\ 0 & 0 & 5 \end{pmatrix}$$

12. [3 pt] Suppose that A is an orthogonal $n \times n$ matrix, and $\mathbf{x}$ and $\mathbf{y}$ are two vectors in $\mathbb{R}^n$. Show that the dot product $(A\mathbf{x}) \cdot (A\mathbf{y})$ is equal to the dot product $\mathbf{x} \cdot \mathbf{y}$.

13. [7 pt] The quadratic form

$$9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$$

is transformed to

$$3y_1^2 + 9y_2^2 + 15y_3^2$$

by an orthogonal change of variables $\mathbf{x} = P\mathbf{y}$, or $\mathbf{y} = P^T\mathbf{x}$.

Find the formulas for y_1, y_2, y_3 in terms of x_1, x_2, x_3.

14. [5 pt] The matrix of observations for two variables over a sample size of 5 individuals is as follows

$$\begin{pmatrix} 1 & 4 & 2 & 6 & 7 \\ 3 & 13 & 6 & 8 & 15 \end{pmatrix}$$

What is the sample covariance matrix of this data set?

15. Two variables x_1, x_2 represent test scores for a group of students. Test score x_1 has a mean of 21 and a variance of 3. The mean of x_2 is 19, and the variance of x_2 is 6. The test scores are positively correlated, and the *covariance* of x_1, x_2 is 2.

   (a) [5 pt] Let $y = c_1x_1 + c_2x_2$ be a composite score, with coefficients $c_1^2 + c_2^2 = 1$. Find the values of c_1, c_2 so that the composite variable y has the largest possible variance.

   (b) [3 pt] What percentage of the total variance of the test scores x_1, x_2 is explained by the composite score y?

TRUE or FALSE?

For each of the statements below indicate whether it is true or false. You do not need to justify your answers. You lose 2 points out of 10 for each answer that is not correct or that is left blank. You cannot lose more than 10 points.

16. [10 pt]

(a) **True** / **False** A linear system of 4 equations in 8 variables is always consistent.

(b) **True** / **False** If a linear transformation $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ is one-to-one, then it must be onto as well.

(c) **True** / **False** If A is an $m \times n$ matrix then the equation $A^T A\mathbf{x} = A^T\mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\mathbb{R}^m$.

(d) **True** / **False** If U is an orthogonal matrix, then $3U$ is invertible and its inverse is $3U^T$.

(e) **True** / **False** A multivariate data set with 400 variables and a sample size of 100 individuals has total sample variation of 200. If the largest eigenvalue of the covariance matrix is $\lambda_1 = 100$, then 25% of the variation in the data can be explained by a single composite variable.

(f) **True** / **False** If a quadratic form $Q(\mathbf{x}) = x^T Ax$ has 3×3 matrix A with eigenvalues $9, 6$ and 0, then $Q(\mathbf{x})$ can never be negative.

(g) **True** / **False** The matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the vector $\mathbf{b}$ is contained in the subspace $\operatorname{Nul} A^T$.

(h) **True** / **False** If $\mathbf{x}$ is a least-squares solution of an inconsistent equation $A\mathbf{x} = \mathbf{b}$, then the vector $A\mathbf{x}$ is the orthogonal projection of $\mathbf{b}$ onto the subspace $\operatorname{Col} A$.

(i) **True** / **False** Every orthogonal set of non-zero vectors is linearly independent.

(j) **True** / **False** If $\mathcal{B}$ is a basis for $\mathbb{R}^n$, and $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$, then the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ of T is a square matrix that is similar to A.

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