Math 22: Linear Algebra with Applications
Professor Rockmore

Final Exam
Sunday, December 7, 2008

No Calculators. Remember the Honor Code - do all of your own work. Take your time and you'll do fine.

Name: Solutions

1.	22 *pts*	
2.	14 *pts*	
3.	8 *pts*	
4.	26 *pts*	
5.	10 *pts*	
6.	14 *pts*	
7.	24 *pts*	
8.	16 *pts*	
9.	20 *pts*	
10.	4 *pts*	
11.	14 *pts*	
12.	3 *pts*	
Total	175 *pts*	

1. **(22 points)** Consider the following system of linear equations

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & + & x_4 & = & 1 \\ -x_1 & - & x_2 & + & 4x_3 & - & x_4 & = & 6 \\ -2x_1 & - & 2x_2 & + & 7x_3 & - & x_4 & = & 1 \end{array}$$

(a) **(2 points)** Put them in the form of a matrix/vector equation $A\vec{x} = \vec{b}$.

$$A = \begin{pmatrix} 1 & 2 & -3 & 1 \\ -1 & -1 & 4 & -1 \\ -2 & -2 & 7 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}.$$

(b) **(6 points)** Either show the system to be inconsistent or find the solutions.

$$\begin{pmatrix} 1 & 2 & -3 & 1 & 1 \\ -1 & -1 & 4 & -1 & 6 \\ -2 & -2 & 7 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 2 & 1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & -1 & 1 & -11 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & -2 & 34 \\ 0 & 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & -1 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -4 & 42 \\ 0 & 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & -1 & 11 \end{pmatrix}$$

$$\begin{cases} x_1 = 4x_4 + 42 \\ x_2 = -x_4 - 4 \\ x_3 = x_4 + 11 \\ x_4 \text{ is free} \end{cases} \qquad X = \begin{pmatrix} 42 \\ -4 \\ 11 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

(c) **(2 points)** If there are solutions, how are the solutions to the associated homogeneous system related to the solutions of the original system.

Solutions of $Ax=b$ are $p+v_h$, where p is a particular solution and v_h is the solution set of $Ax=0$.

So $x_4 \begin{pmatrix} 4 \\ -1 \\ 1 \\ 1 \end{pmatrix}$ are all the solutions to $Ax=0$

(d) **(2 points)** What is $dim(Null(A))$?

There is one free variable, so $dim(Nul(A)) = 1$

(e) **(2 points)** Define what is meant by the "rank of a matrix."

It's $dim(Col(A))$

(f) **(2 points)** What is $rank(A)$?

There are 3 pivot columns, so $rank(A) = 3$

(g) **(2 points)** What is $rank(A)$?

? Same as (f) ?

(h) **(2 points)** What is $dim(range(A))$?

NCX

$dim(range(A)) = rank(A) = 3$

(i) **(2 points)** What is $dim(row(A))$?

NCX

$dim(row(A)) = rank(A) = 3$

2. **(14 points)**

(a) **(3 points)** Let vectors $\vec{v_1}, \ldots, \vec{v_n}$ be in vector space V. What does it mean for them to be linearly dependent?

Many ways to say it, e.g. if $A = (v_1 \cdots v_n)$ then $Ax = 0$ only has the trivial solution.

(b) **(3 points)** Let $T : V \longrightarrow W$ be a linear transformation between real vector spaces V and W. What does it mean for T to be linear?

$$T(u+v) = T(u) + T(v)$$
$$T(cu) = cT(u)$$

(c) **(4 points)** Suppose A is an $n \times n$ real matrix. Give two different conditions for A to be nonsingular (i.e., invertible).

Pick any two. $\det(A) \neq 0$

A is row-equivalent to I_n

(d) **(4 points)** Let $\vec{b} \in \mathbf{R}^n$ such that the linear system $A\vec{x} = \vec{b}$ is inconsistent.

What do we mean by the statement "$\vec{v} \in \mathbf{R}^m$ is a least squares solution to the system $A\vec{x} = \vec{b}$?" Give an "analytic" answer (i.e., a mathematical statement) as well as a "geometric" definition. (Hint: The latter should involve $Span(Col(A))$.)

NET

To find the solution we solve $A^T A x = A^T b$.

Geometrically we project b onto the column space of A, so that $Ax = \hat{b}$ is consistent, and solve.

3. **(8 points)**

NEX The vectors

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

form a basis for $\mathbf{R}^3$. All vectors are written with respect to the standard basis for $\mathbf{R}^3$.

(a) **(4 points)** Let $\vec{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ be a vector written with respect to the standard basis. Give a matrix/vector expression for the coordinates of $\vec{v}$ with respect to the basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. **NOTE:** If your expression includes the inverse of some matrix, you need not actually invert the matrix.

While you could solve it by reducing $\left(v_1 \; v_2 \; v_3 \mid v \right)$, the problem only wants you to say

$$[x]_B = P_B^{-1} x \quad \text{where} \quad P_B = (v_1 \; v_2 \; v_3) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

(b) **(4 points)** Suppose A is a 3×3 real matrix representing a transformation of $\mathbf{R}^3$ relative to the standard basis. What is the matrix that expresses the transformation relative to the basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. **NOTE:** If your expression includes the inverse of some matrix, you need not actually invert the matrix.

As above, the problem only wants you to say that, since $A = {}_{\mathcal{E}}[T]_{\mathcal{E}}$ where $\mathcal{E}$ is the standard basis

$${}_B[T]_B = P_{B \leftarrow \mathcal{E}} \; {}_{\mathcal{E}}[T]_{\mathcal{E}} \; P_{\mathcal{E} \leftarrow B} = P_B^{-1} A P_B \quad \text{w/ } P_B \text{ as above.}$$

4. **(26 points)**

(a) **(4 points)** Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & -1 \\ 4 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

$$1 \cdot \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} = 6 - 16 - 1 = -11$$

(b) Let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

i. **(2 points)** Compute the determinant of A.

$$3 - 8 = -5$$

ii. **(2 points)** Does A^{-1} exist? If so, what is it?

Yes, $\det(A) \neq 0$ $\quad A^{-1} = \frac{1}{-5} \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}$

iii. **(2 points)** What is the area of image of the unit square in $\mathbf{R}^2$ (it will be some parallelogram) under the transformation A.

Area (unit square) $= 1$, Area (parall) $= |\det(A)| \cdot 1 = 5$

iv. **(2 points)** Compute the characteristic equation for A.

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$$

v. **(6 points)** The matrix A is diagonalizable - find its eigenvalues and the corresponding eigenvectors.

As above, $\lambda = 5, -1$

$$A - 5I = \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A + I = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

NCt vi. **(2 points)** Give a geometric interpretation for what A does to $\mathbf{R}^2$. (Hint: use the fact that the eigenvectors form a basis for $\mathbf{R}^2$).

A stretches vectors in the v_1 direction by a factor of 5 and vectors in the v_2 direction by a factor of -1

NCt vii. **(2 points)** Suppose B is a 2×2 matrix that effects a rotation in the plane ($\mathbf{R}^2$). Will B have real eigenvectors and real eigenvalues? Why or why not? (Hint: Think geometrically what it means to be an eigenvector!).

Because B rotates but does not stretch, it won't have real eigenvalues

viii. **(2 points)** Give an example of a 2×2 rotation matrix.

General rotation $= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\theta = \frac{\pi}{2})$

NCt ix. **(2 points)** What are the eigenvalues of the rotation matrix you gave above?

$\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 \Rightarrow \lambda = \pm i$

5. **(10 points)**

(a) **(2 points)** Let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

Write down a matrix that does not commute with A.

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

NCY (b) **(2 points)** Suppose that $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a nonzero 2×2 matrix (so it has at least one nonzero entry). Show that there is some vector $\vec{v} \in \mathbf{R}^2$ such that $B\vec{v} \neq \vec{0}$.

Suppose $a \neq 0$. Then $B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \neq 0$.

NCY (c) **(6 points)** Suppose that B and C are 2×2 matrices with the same eigenvectors $\vec{v}, \vec{w}$ and that $\vec{v}$ and $\vec{w}$ are a basis for $\mathbf{R}^2$. I.e.,

$$B\vec{v} = \lambda_1\vec{v} \quad B\vec{w} = \lambda_2\vec{w}; \quad C\vec{v} = \mu_1\vec{v} \quad C\vec{w} = \mu_2\vec{w}$$

Show that A and B commute - i.e., $AB = BA$. (Hint: consider what they do to an arbitrary vector $\vec{u} \in \mathbf{R}^2$.

$u = c_1 v + c_2 w$ $\quad Bu = B(c_1 v + c_2 w) = c_1 \lambda_1 v + c_2 \lambda_2 w$

$CBu = c_1 \lambda_1 \mu_1 v + c_2 \lambda_2 \mu_2 w$

Check that the same is true for BCu.

6. **(14 points)**

Let

$$\vec{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 7 \end{pmatrix}$$

(a) **(2 points)** What is the length of $\vec{u}$?

$$\sqrt{1^2 + (-2)^2 + 3^2 + 0^2} = \sqrt{14}$$

NET

(b) **(2 points)** What is the cosine of the angle between $\vec{u}$ and $\vec{v}$?

$$\cos\theta = \frac{U \cdot V}{\|U\|\|V\|} = \frac{-7}{\sqrt{14}\sqrt{59}}$$

(c) **(2 points)** What is the distance between the points in $\mathbf{R}^4$ represented by $\vec{u}$ and $\vec{v}$.

$$\|u - v\| = \left\| \begin{pmatrix} 2 \\ -5 \\ 3 \\ -7 \end{pmatrix} \right\| = \sqrt{4 + 25 + 9 + 49} = \sqrt{87}$$

(d) **(2 points)** Compute the projection of $\vec{v}$ onto $\vec{u}$.

$$\text{proj}_u v = \frac{v \cdot u}{u \cdot u} u = \frac{-7}{14} u = \begin{pmatrix} -1/2 \\ 1 \\ -3/2 \\ 0 \end{pmatrix}$$

NET (e) **(4 points)** Write down the matrix that computes that projection of any vector $\vec{w} \in \mathbf{R}^4$ onto $\vec{u}$.

As in 7.1, it's $\frac{1}{\|u_1\|^2} u_1 u_1^T = \frac{1}{14} \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 & 0 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 1 & -2 & 3 & 0 \\ -2 & 4 & -6 & 0 \\ 3 & -6 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(f) **(2 points)** Use (c) to write $\vec{v}$ as a $\vec{v} = \vec{w_1} + \vec{w_2}$ where $\vec{w_1} \in Span(\vec{u})$ and $\vec{w_2} \in (Span(\vec{u}))^{\perp}$.

I think it means use (d).

$$V = \underbrace{\text{proj}_u v}_{\in\, \text{Span}(u)} + \underbrace{(v - \text{proj}_u v)}_{\in\, \text{Span}(u)^{\perp}} = \begin{pmatrix} -1/2 \\ 1 \\ -3/2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 2 \\ 3/2 \\ 7 \end{pmatrix}$$

Alternatively, compute $\text{proj}_u e_1$, $\text{proj}_u e_2$, $\text{proj}_u e_3$ and $\text{proj}_u e_4$ and stick them in a matrix.

7. **(24 points)**

Suppose A is a 3×3 matrix with orthogonal eigenvectors

$$\vec{u}_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

and eigenvalues $1, 0, -\frac{2}{3}$ respectively.

(a) **(6 points)** Give the spectral decomposition for A.

$$A = 1 \cdot u_1 u_1^T + 0\, u_2 u_2^T + \frac{2}{3}\, u_3 u_3^T$$

This would be enough

(b) **(4 points)** Is A symmetric? Why or why not?

Yes, because $u_1 u_1^T$ are symmetric matrices

(c) Let $\vec{v} = 6\vec{u}_1 - 2\vec{u}_2 + 3\vec{u}_3$

i. **(4 points)** Compute the projection of $\vec{v}$ onto the subspace spanned by $\vec{u_1}$ and $\vec{u_2}$.

Because the u_1's are orthogonal and $v = 6u_1 - 2u_2 + 3u_3$,

$\hat{v} = 6u_1 - 2u_3$ $\left[\frac{v \cdot u_1}{u_1 \cdot u_1} = 6, \quad \frac{v \cdot u_2}{u_2 \cdot u_2} = -2 \right]$

ii. **(2 points)** Let $W = Span\{\vec{u_1}, \vec{u_2}\}$. What is the vector in W that is closest to $\vec{v}$?

It's $v - proj_W v = v - \hat{v} = 3u_3$

NEX

iii. **(4 points)** Write down the matrix that takes as input a vector $\vec{w}$ (in standard coordinates) and computes its projection onto the subspace spanned by $\vec{u_1}$ and $\vec{u_2}$.

$u_1 u_1^T + u_2 u_2^T$

iv. **(2 points)** Compute Av (in terms of $\vec{u_1}, \vec{u_2}, \vec{u_3}$).

$Au_1 = u_1$, $Au_2 = 0$, $Au_3 = -\frac{2}{3} u_3$ $\Longrightarrow$ $Av = 6u_1 + 3 \cdot \left(-\frac{2}{3}\right) u_3 =$

$= 6u_1 - 2u_3$

NEX

v. **(2 points)** What is $\lim_{n \to \infty} A^n \vec{v}$?

Since $\left(-\frac{2}{3}\right)^n \to 0$ as $n \to \infty$, it's $6u_1$

Partially NEY

8. **(16 points)**
Let

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{pmatrix}$$

(a) **(8 points) Use the Gram-Schmidt process to find an orthogonal basis for $Col(A)$.**

You'd find a basis for col(A) as usual

$$\begin{pmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \\ 0 & -4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$ all the columns are in the basis.

Then you use Gram-Schmidt (see section 6.4) to orthogonalize it

$$V_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix} \qquad V_2 = \begin{pmatrix} -2 \\ 0 \\ -4 \\ 0 \end{pmatrix} - \text{proj}_{v_1} \begin{pmatrix} -2 \\ 0 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -4 \\ 0 \end{pmatrix} - \frac{-10}{25} \begin{pmatrix} -2 \\ 0 \\ -4 \\ 0 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} - \text{proj}_{v_1} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} - \text{proj}_{v_2} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

NET

(b) (8 points) Find the QR decomposition of A.

See section 6.4

Given v_1, v_2, v_3 as in part (a) normalize them to an orthonormal basis u_1, u_2, u_3

Then $Q = (u_1\ u_2\ u_3)$ and $R = Q^T A$

9. **(20 points)**

Suppose that the following table denotes a stock price at times $t_i = 0, 1, 2, 3$.

t_i	0	1	2	3
y_i	1	2	1	5

(a) **(8 points)** Set up and solve the system of normal equations to find the equation of the straight line that best approximates (in a least squares sense) the data.

$$y = \beta_0 + \beta_1 t \implies X\beta = y \quad \text{with}$$

$$X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \quad y = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \end{pmatrix}$$

$$X^T X = \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \quad X^T y = \begin{pmatrix} 9 \\ 19 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 & 9 \\ 6 & 14 & 19 \end{pmatrix} \sim \begin{pmatrix} 4 & 6 & 9 \\ 2 & 8 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 5 \\ 4 & 6 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 5 \\ 0 & -10 & -11 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 6/10 \\ 0 & 1 & 11/10 \end{pmatrix} \qquad y = \frac{6}{10} + \frac{11}{10} t$$

(b) **(2 points)** Suppose that the data (t, y) represents the price (y) of a stock at time t. What would be the (linear) prediction of the stock price at time $t = 5$?

$$y = \frac{6}{10} + \frac{11}{10} t \longrightarrow \frac{6}{10} + \frac{11}{10} \cdot 5 = \frac{61}{10}$$

(c) Suppose we want to find the best quadratic (i.e., second degree) polynomial approximation (in a least squares sense) to the data. I.e., the "best" polynomial $y(t) = \beta_0 + \beta_1 t + \beta_2 t^2$ to approximate the data.

i. **(6 points)** Pose this as a least squares problem – i.e., give the design matrix for solving the associated least squares problem.

ii. **(4 points)** Give the associated collection of normal equations – you can present this as a matrix/vector system of equations. DO NOT SOLVE THEM.

$$X\beta = y \quad X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \quad y \text{ as above}$$

The eqns are $X^T X \beta = X^T y$

$$X^T X = \begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 37 \\ 14 & 37 & 98 \end{pmatrix} \quad X^T y = \begin{pmatrix} 9 \\ 19 \\ 51 \end{pmatrix}$$

10. **(4 points)**

Write down the design matrix for finding the best approximation (in a least squares sense) by a plane $z = \beta_0 + \beta_1 x + \beta_2 y$ for the data

x_i	1	2	3	4
y_i	1	1	-1	0
z_i	3	-2	8	4

I.e., The value z_i is what was observed for a given input of (x_i, y_i).

$$X\beta = z \qquad X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \\ 1 & 4 & 0 \end{pmatrix} \qquad z = \begin{pmatrix} 3 \\ -2 \\ 8 \\ 4 \end{pmatrix}$$

It only wants the design matrix, so that's it

NEX 11. **(14 points)**

(a) **(2 points)** Suppose A represents the matrix of a regular Markov chain. What is the equilibrium distribution and how do you find it?

The equilibrium distribution or steady state is the unit vector q such that $Aq = q$.

(b) Let A be the matrix of a discrete dynamical with eigenvalues $\frac{1}{4}$ and $\frac{1}{2}$ with eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

i. **(2 points)** Classify the origin as an attractor, repellor or saddle point.

Both eigenvalues are < 1, so it's an attractor

ii. **(2 points)** In what direction does the trajectory change the fastest?

It's the direction whose eigenvalue is the smallest, so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

iii. **(4 points)** If $\vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, what is $\vec{x}_1 = A\vec{x}_0$?

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $Ax_0 = \frac{1}{2} A\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2} A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\cdot\frac{1}{4}\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\cdot\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

iv. **(4 points)** For a positive integer k, what is $\vec{x}_k = A^k\vec{x}_0$?

$x_k = \frac{1}{2}\left(\frac{1}{4}\right)^k\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}\cdot\left(\frac{1}{2}\right)^k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$