Math 22, Exam II

May 13, 2010

NAME:

This is a closed book exam and you may not use a calculator. Use the space provided to answer the questions and if you need more space, please use the back of the exam making sure to write a note in the space provided that you have more work elsewhere that you would like me to grade. You must SHOW ALL WORK and be neat. If you have any questions, do not hesitate to ask.

Good luck!

Remember the honor code – do all of your own work.

1. The matrix A has been converted to echelon form as follows:

$$A = \begin{pmatrix} -20 & -59 & -97 & 120 & -219 & -225 \\ 1 & 4 & 8 & -6 & 12 & 48 \\ 1 & 4 & 8 & -6 & 54 & 27 \\ 1 & 4 & 29 & -111 & 96 & 90 \\ 1 & 25 & 29 & 204 & -261 & 132 \\ 22 & 46 & 71 & -237 & 390 & 90 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & -6 & 11 & 12 \\ 0 & 1 & 2 & 5 & -7 & 9 \\ 0 & 0 & 1 & -5 & 6 & 4 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 19 & 0 & 0 \\ 0 & 1 & 0 & 15 & 0 & 0 \\ 0 & 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

5

$$\begin{pmatrix} 1 & 0 & 0 & -26 & 0 & 0 \\ 0 & 1 & 0 & 15 & 0 & 0 \\ 0 & 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a. Write down a basis for the row space of A.

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \\ -6 \\ 11 \\ 12 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \\ -7 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -5 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

b. Write down a basis for the column space of A.

$$\left\{ \begin{bmatrix} -20 \\ 1 \\ 1 \\ 1 \\ 1 \\ 22 \end{bmatrix}, \begin{bmatrix} -59 \\ 4 \\ 4 \\ 4 \\ 25 \\ 46 \end{bmatrix}, \begin{bmatrix} -97 \\ 8 \\ 8 \\ 29 \\ +29 \\ 71 \end{bmatrix}, \begin{bmatrix} -219 \\ 12 \\ 54 \\ 96 \\ -261 \\ 390 \end{bmatrix}, \begin{bmatrix} -225 \\ 48 \\ 27 \\ 90 \\ 132 \\ 90 \end{bmatrix} \right\}$$

c. What is the dimension of Col(A)? 5

d. Write down a basis for the null space of A.

$$\left\{ \begin{bmatrix} 26 \\ -15 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

e. What is the dimension of the subspace of all solutions $\mathbf{x}$ of $A^T\mathbf{x} = \mathbf{0}$?

$\text{rank } A = \text{rank } A^T = 5 \Rightarrow$

$\dim \text{Null } A^T = 6 - 5 = \underline{1}$

2. Let

$$A = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}$$

and let

$$\mathbf{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

a. Show that **v** and **w** are eigenvectors for A with eigenvalues 5 and 1, respectively.

1). $A\bar{v} = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \end{pmatrix} = 5\begin{pmatrix} -2 \\ 1 \end{pmatrix} = 5\bar{v}$;

2). $A\bar{w} = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \bar{w}$.

b. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$$P = [\bar{v} \;\; \bar{w}] = \begin{bmatrix} -2 & -2 \\ 1 & 3 \end{bmatrix}.$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$P^{-1} = \begin{bmatrix} -2 & -2 \\ 1 & 3 \end{bmatrix}^{-1} = -\frac{1}{4}\begin{bmatrix} 3 & 2 \\ -1 & -2 \end{bmatrix} \qquad (\det P = -4)$$

$$= \begin{bmatrix} -3/4 & -1/2 \\ 1/4 & 1/2 \end{bmatrix}.$$

Check $PDP^{-1} = \begin{pmatrix} -2 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3/4 & -1/2 \\ 1/4 & 1/2 \end{pmatrix}$

$$= \begin{pmatrix} -10 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} -3/4 & -1/2 \\ 1/4 & +1/2 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix} = A$$

3. Let A be a 3×3 matrix whose eigenvectors are

$$\bar{v} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \quad \bar{w} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

of eigenvalues 1, -1 and 2 respectively. Find A.

$$P = [\bar{v} \ \ \bar{w} \ \ \bar{x}] = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix}; \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 3/8 & 3/4 & -1/8 \\ 1/4 & 1/2 & 1/4 \\ -3/8 & 1/4 & 1/8 \end{bmatrix}$$

So $$PDP^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3/8 & 3/4 & -1/8 \\ 1/4 & 1/2 & 1/4 \\ -3/8 & 1/4 & 1/8 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 0 & -1 & -4 \\ 1 & 0 & 2 \\ -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 6 & -1 \\ 2 & 4 & 2 \\ -3 & 2 & 1 \end{bmatrix} =$$

$$= \frac{1}{8} \begin{bmatrix} 10 & -12 & -6 \\ -3 & 10 & 1 \\ -12 & -24 & -4 \end{bmatrix} = \begin{bmatrix} 5/4 & -3/2 & -3/4 \\ -3/8 & 5/4 & 1/8 \\ -3/2 & -3 & -1/2 \end{bmatrix}.$$

4. Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

a. What are the eigenvalues of A?

1 and 2

b. What are the algebraic multiplicities of each eigenvalue?

1 with multiplicity 2 and 2 with multiplicity 1.

c. What are the geometric multiplicities of each eigenvalue?

$\lambda = 1$; $A - I_3 = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ - one free variable, so $\underline{1}$

$\lambda = 2$ - multiplicity 1

d. Is A diagonalizable?

No! The geometric multiplicity of $\lambda = 1$ is smaller than its algebraic multiplicity.

5. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}.$$

Observe that $\mathcal{B}$ is a basis for $\mathbf{R}^2$. For $\mathbf{x} = \begin{pmatrix} -7 \\ 8 \end{pmatrix}$ compute $[\mathbf{x}]_{\mathcal{B}}$.

$$\bar{x} = P_B [\bar{x}]_B \quad \Rightarrow \quad [x]_B = P_B^{-1} \bar{x}.$$

$$P_B = [\bar{b}_1 \ \bar{b}_2] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \quad \det P_B = -3$$

$$P_B^{-1} = -\frac{1}{3} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}$$

$$[\bar{x}]_B = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} -7 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

6. The polynomials $\mathcal{B} = \{1, t-2, (t+2)^2\}$ form a basis for $\mathbb{P}_2$. For $\mathbf{x} = 1 + t + t^2$ find $[\mathbf{x}]_{\mathcal{B}}$.

$$1 + t + t^2 = (t^2 + 4t + 4 - 4t - 4) + t + 1 =$$

$$= (t+2)^2 - 3t - 3 = (t+2)^2 - 3[(t-2) + 2] - 3$$

$$= (t+2)^2 - 3(t-2) - 9.$$

$$[\bar{x}]_B = \begin{bmatrix} -9 \\ -3 \\ 1 \end{bmatrix}.$$

Or, $P_B = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Now $P_B[\bar{x}]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ his augmented matrix

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$ Answer is again

$$\begin{bmatrix} -9 \\ -3 \\ 1 \end{bmatrix}.$$

7. Compute the characteristic polynomials of the following matrices.

a.

$$A = \begin{pmatrix} 2 & -2 \\ 1 & 5 \end{pmatrix}.$$

$$\det(A - \lambda I_2) = \begin{vmatrix} 2-\lambda & -2 \\ 1 & 5-\lambda \end{vmatrix} = (\lambda-2)(\lambda-5) + 2$$
$$= \lambda^2 - 7\lambda + 12 = (\lambda-3)(\lambda-4)$$

b.

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$\det(B - \lambda I_3) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 1)$$
$$= -\lambda^3 + 2\lambda^2 + \lambda - 2$$

c. For the matrix A, what are the eigenvalues?

3 and 4

d. For the matrix B, what are the eigenvalues?

$\lambda = \pm 1, 2.$

8. True or false:

a. The only eigenvalue of the **0** matrix is 0.

$\bar{0} = \bar{0}\bar{x} = \lambda \bar{x}$, $\bar{x} \neq \bar{0}$ $\Rightarrow$ $\lambda = 0$ TRUE.

b. 7 is an eigenvalue of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 7 \\ 0 & 0 & 1 \end{pmatrix}.$$

FALSE. EIGENVALUES ARE $\lambda = 1$ (multiplicity 2)
$\lambda = 3$.

c. The sum of two diagonal matrices is a diagonal matrix.

YES!

d. The set of polynomials of the form $2t - at^2 + bt^3$, where a and b are arbitrary real numbers is a subspace of $\mathbb{P}_3$.

NO! THE 0 POLYNOMIAL DOES NOT HAVE 2 AS THE COEFFICENT OF T.

e. If A is a 7×8 matrix having rank 4, then its null space is 4 dimensional.

$8 = \text{rank } A + \dim \text{Null } A \Rightarrow \dim \text{Null } A = 4.$ TRUE.

f. A matrix A having distinct eigenvalues is invertible.

FALSE! $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ HAS DISTINCT EIGENVALUES (0 AND 1) AND IT IS NOT INVERTIBLE.

9. Show that if λ is an eigenvalue for A, then 2λ is an eigenvalue for $2A$.

Say $\bar{x}$ is non zero and $A\bar{x} = \lambda \bar{x} \Rightarrow$

$2(A\bar{x}) = 2(\lambda \bar{x})$ or

$(2A)\bar{x} = (2\lambda)\bar{x}$.

So, 2λ is an eigen value for $2A$.

10. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}, \quad \text{and} \quad \mathcal{C} = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Find the change of basis that converts an element in $\mathcal{B}$ coordinates to an element in $\mathcal{C}$ coordinates (usually denoted by $P_{\mathcal{C}\leftarrow\mathcal{B}}$).

$P_{C\leftarrow B}$ to get it start with

$$\left[\bar{c}_1 \ \bar{c}_2 \ \vdots \ \bar{b}_1 \ \bar{b}_2 \right] = \left[\begin{array}{cc|cc} -1 & 1 & 5 & 2 \\ 0 & 1 & 3 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cc|cc} 1 & -1 & -5 & -2 \\ 0 & 1 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

So $\underset{C\leftarrow B}{P} = \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}.$