Definition 1. (i) Let $W_1, W_2, \ldots, W_k$ be subspaces of $\mathbb{R}^n$ for some n. The *sum* of the W_i is

$$\sum_{i=1}^{k} W_i = \{w_1 + w_2 + \ldots + w_k : w_i \in W_i 1 \leq i \leq k\}.$$

(ii) $\mathbb{R}^n$ is the *direct sum* of subspaces $W_1, W_2, \ldots, W_k$ if their sum is $\mathbb{R}^n$ and their pairwise intersection is the zero vector (i.e., $W_i \cap W_j = \{\mathbf{0}\}$ for all $i \neq j$).

Lemma 2. *Suppose $W_1, W_2, \ldots, W_k$ are subspaces of $\mathbb{R}^n$ such that their sum is $\mathbb{R}^n$. Then if $X_1, X_2, \ldots, X_k$ are chosen such that for each i, $W_i = \mathrm{Span}(X_i)$, $\bigcup X_i$ spans $\mathbb{R}^n$.*

Lemma 3. *Suppose $W_1, W_2, \ldots, W_k$ are subspaces of $\mathbb{R}^n$ such that $\mathbb{R}^n$ is their direct sum. Then if $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$ are vectors such that each $\boldsymbol{w}_i$ is an element of W_i and not all the $\boldsymbol{w}_i$ are $\mathbf{0}$, the sum $\boldsymbol{w}_1 + \ldots + \boldsymbol{w}_k$ is also not $\mathbf{0}$. In particular, any set of nonzero vectors, each of which is drawn from a distinct subspace W_i, is linearly independent.*

Theorem 4. *Let $W_1, W_2, \ldots, W_k$ be subspaces of $\mathbb{R}^n$ that sum to $\mathbb{R}^n$. Then $\mathbb{R}^n$ is the direct sum of $W_1, W_2, \ldots, W_k$ if and only if the sum of the dimensions of the W_i is n.*

Proof.