Your name:

Instructor (please circle): Alex Barnett Michael Musty

Math 22 Summer 2017, Midterm 2, Tues Aug 8

Please show your work. No credit is given for solutions without work or justification.

1. [6 points]

   (a) Compute the determinant of $\begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & -1 & 0 & 2 \\ 2 & 21 & 7 & -4 \\ 0 & 3 & 1 & 5 \end{bmatrix}$.

   (b) Let A_0 be an invertible 4×4 matrix with $\det A_0 = 1$, and suppose:

   - A_1 is obtained from A_0 by interchanging 2 rows,
   - A_2 is obtained from A_1 (note: not A_0) by scaling a row of A_1 by 3,
   - A_3 is obtained from A_2 (note: not A_0) by row-replacement.

   Find the determinants of these matrices and fill them in below:

$$\det A_1 =$$
$$\det A_2 =$$
$$\det A_3 =$$

2. [8 points] Let $A = \begin{bmatrix} -1 & 2 & -6 & -3 \\ 2 & -4 & 7 & 6 \\ -1 & 2 & -3 & -3 \end{bmatrix}$ which is row-equivalent to $\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for Col A.

(b) Find a basis for Nul A.

(c) Prove that the null space of any 3×4 matrix A is a subspace of $\mathbb{R}^4$.

3. [6 points]

(a) Let $H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\}$.

Is the set $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ a basis for H? Explain.

(b) Let A and B be matrices such that AB exists. Prove that rank $(AB) \leq$ rank A. [Hint: each column of AB is in Col A.]

4. [9 points] Let $A = \begin{bmatrix} 5 & -4 & -2 \\ 2 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$.

[Hint: numbers will come out very simply, so stop and check your work if they are not!]

(a) Use the characteristic polynomial to find A's eigenvalues and their algebraic multiplicies:

(b) For each distinct eigenvalue of A, find a basis for its eigenspace:

(c) Evaluate $A^{2017} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

5. [8 points]

(a) A linear system has a system matrix A of size 7×9 (ie 7 equations in 9 unknowns). Say you know that there is some right-hand side vector for which there is no solution. What is the smallest dim Nul A may be, and why?

(b) Now let A be any matrix. If the system $A\mathbf{x} = \mathbf{b}$ is consistent for all right-hand sides $\mathbf{b}$, explain why the system $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) Let A be any matrix. Is some subset of the rows of a A a basis for Row A ? Prove your answer. [As always, indicate what if any theorem(s) you use.]

6. [6 points]

(a) Is the set $\{1+t, 1-t, t+2t^2\}$ a basis for $\mathbb{P}_2$? Prove your answer. [State any theorems or results that you use.]

(b) Find the coefficients of $4(1+t)^2$ relative to the set from part (a).

7. [7 points] In this question only, no working is needed; just circle T or F.

(a) T / F: The eigenvalues of a lower-triangular matrix (ie, all zeros above the diagonal) are the diagonal entries.

(b) T / F: The dimension of an eigenspace can never exceed the algebraic multiplicity of the corresponding eigenvalue.

(c) T / F: An eigenvector with eigenvalue 2 could be a linear combination of an eigenvector with eigenvalue 3 and an eigenvector with eigenvalue 4.

(d) T / F: Row reduction of a matrix always preserves its row space.

(e) T / F: Row reduction of a square matrix always preserves its eigenvalues.

(f) T / F: If $T : \mathbb{R}^5 \to \mathbb{R}^5$ is a linear transformation with standard matrix A and the rank of A is 2, then it is possible to have $T(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x}$ in the domain.

(g) T / F: Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is an isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^m$, with $n, m > 0$, with standard matrix A. Then it is impossible for Nul A and Col A to have the same dimension.