Your name:

Instructor (please circle): Alex Barnett Naomi Tanabe

Math 22 Fall 2016, Final, Fri Nov 18 (80 points total)

Please show your work. No credit is given for solutions without work or justification.

1. [10 points]

   (a) Is the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ diagonalizable?

   (b) Is the matrix $\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$ diagonalizable?

   (c) Let A be whichever of the above matrices was diagonalizable. Give a matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(d) Let A be the same matrix as in (c). Give an expression (involving only numbers) for the result when the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is multiplied by A from the left 2016 times.

2. [11 points]

(a) Suppose the web consists of two pages: page a links to page b (that's it!) Use the PageRank algorithm (with $\alpha = 1$) to compute the vector $\mathbf{q}$ of importance scores. Scale your answer so that it is a probability vector:

(b) As you know, in the real world the number n of web pages exceeds 10^{10}, making the above linear solve impossibly expensive. Explain in one sentence how Google in practice approximates $\mathbf{q}$.

(c) Give the definition of a $n \times n$ matrix being *stochastic.*

(d) Prove that any $n \times n$ stochastic matrix A has an eigenvalue of 1.

BONUS Prove that the procedure in (b) always works for your matrix from (a). Estimate how long it would take to get to 1% accuracy in $\mathbf{q}$.

3. [10 points] Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) What is the maximum $\|A\mathbf{x}\|$ can have over all of the unit vectors $\|\mathbf{x}\| = 1$?

(b) Compute the *full* singular value decomposition of A (i.e. give U, Σ and V. Choose signs so that the top row of V has positive entries.)

(c) Which row(s) or column(s) of which matrix gives an orthonormal basis for Nul (A^T) ?

4. [10 points] Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$.

(a) Is $\mathbf{b}$ in Col A ?

(b) Find the complete set of least squares solutions to $A\mathbf{x} = \mathbf{b}$.

(c) Write $\mathbf{b}$ as the sum of some vector in Col A and some vector orthogonal to Col A.

(d) Is your answer to (c) unique? Why?

5. [9 points] Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 4 \end{bmatrix}$.

(a) Find an orthogonal basis for Col A.

(b) Compute the QR decomposition of A.

(c) Use your previous answer to write a formula for the matrix that maps any point $\mathbf{b}$ in $\mathbb{R}^3$ to its *nearest* point in Col A. [You can leave it as an expression; don't write out matrix elements . . . unless you enjoy pain]

6. [10 points] Short ones.

(a) Let $A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & a \\ a & 2 & 3 \end{bmatrix}$ with a real number a. For what values of a is the matrix A invertible?

(b) Suppose that a general matrix A has linearly independent columns. Prove that PA, for any invertible matrix P, also has linearly independent columns. [Hint: if stuck first prove the converse.]

(c) A matrix A such that $A^T A = AA^T$ is called "normal". Prove that for any normal matrix and any $\mathbf{x}$, the length of $A\mathbf{x}$ equals the length of $A^T\mathbf{x}$.

BONUS Prove the amazing fact that, for any square matrix, the sum of its eigenvalues equals the sum of its diagonal entries (its "trace"). You will need to use the back of the page.

7. [10 points] Shorter ones.

(a) Let $W = \left\{ \begin{bmatrix} a \\ a \end{bmatrix}, \ a \text{ in } \mathbb{R} \right\}$. Prove whether or not W is a vector space.

(b) With W as above, every point in W is in the span of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Is the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for W? Explain.

(c) Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ a basis for the set $W = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \right\}$? Explain.

8. [10 points] In this question only, no working is needed; just circle T or F.

(a) T / F: The subset of $\mathbb{P}_2$ defined by $V = \{at^2 + bt + c \ : \ a, b, c \text{ real}, a \neq 0\}$, ie polynomials of degree precisely 2, is a subspace of $\mathbb{P}_2$.

(b) T / F: A 3×5 linear system can have a unique solution.

(c) T / F: If W is a subspace of $\mathbb{R}^n$, then every point in $\mathbb{R}^n$ is either in W or in $W^\perp$.

(d) T / F: Every symmetric $n \times n$ matrix has a set of eigenvectors which may be chosen to be an orthonormal basis for $\mathbb{R}^n$.

(e) T / F: Every square matrix which is diagonalizable is invertible.

(f) T / F: If A is an orthogonal square matrix, A^3 must also be orthogonal.

(g) T / F: The set of all unit vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

(h) T / F: The set of 3-component vectors whose entries sum to zero is isomorphic to $\mathbb{R}^2$.

(i) T / F: If $A = QR$ is the QR decomposition of a square matrix, then the eigenvalues of RQ are identical to those of A.

(j) T / F: For any $m \times n$ matrix, rank A^T = rank A.