Math 20, Test 1, April 21, 2015

Instructions. Show your work and explain carefully. Calculators and other electronic aids are not permitted. (Arithmetic mistakes will not be judged too harshly.) Each problem is worth 10 points.

1. In a race with 5 hourses with odds set fairly, the favorite is Flea Biscuit with odds of 2 : 3 to win, while Tick Muffin has odds of 1 : 4 to win. What are the odds for the event that either Flea Biscuit or Tick Muffin wins?

   The probability for Flea to win is 2/5 and the probability for Tick to win is 1/5. Thus, the probability that one of them wins is 3/5, which is equivalent to 3 : 2 odds.

2. A fair coin is tossed repeatedly till it comes up heads. What is the probability that this occurs on the second toss?

   You're asked for the probability of the sequence TH, which is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

3. Three cards are chosen at random from a standard deck of 52 cards. What is the probability that 1 of the 3 is the ace of spades?

   Solution 1: The sample space has size $\binom{52}{3}$. The event space has size $\binom{51}{2}$ corresponding to the 2 cards going along with the ace of spades. So, the probability is the quotient of these two, which reduces to 3/52.

   Solution 2: The complementary event has size $\binom{51}{3}$, so the probability is $1 - \binom{51}{3}/\binom{52}{3}$.

   Solution 3: The probability that the ace of spades is chosen with the first draw is 1/52. The probability it is not chosen on the first draw, but is chosen on the second is

   $$\frac{51}{52} \cdot \frac{1}{51} = \frac{1}{52}.$$

   Similarly, the probability that it is not chosen on the first two draws, but it is on the third, is

   $$\frac{51}{52} \cdot \frac{50}{51} \cdot \frac{1}{50} = \frac{1}{52}.$$

   Adding the three cases we get 3/52.

4. A card is picked at random from a standard deck of 52 cards, then replaced, and the deck shuffled. After 52 picks, what is the approximate probability that the ace of spades is chosen at least once?

   The exact probability is $1 - (51/52)^{52}$ (the sample space has size 52^{52}, while the complementary event has size 51^{52}). We learned that

   $$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e},$$

   so an approximate answer is $1 - 1/e$.

5. The fourth row of Pascal's triangle is 1 4 6 4 1, and note that the alternating sum $1-4+6-4+1=0$. Show this is no accident: the alternating sum of each row of Pascal's triangle, starting with the first row (which is 1 1), is always 0.

   Solution 1: You're being asked to prove that for $n \geq 1$ that

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0.$$

   But this sum is exactly what you'd get if you expanded $(-1+1)^n$ via the binomial theorem, and evidently this last expression is 0.

   Solution 2: This is a combinatorial proof, and it begins with the fact that the sum of all the entries on row n of Pascal's triangle is 2^n. This follows from the fact that this sum represents the total number of subsets of an n-element set, and for a generic subset, each of the n elements has a choice: are you in or not? There are n such independent choices being made, so there are 2^n different subsets. Now for the problem. Consider row n and consider those subsets of $\{1, 2, \ldots, n\}$ with an even number of entries. The number of these subsets corresponds to the sum of $\binom{n}{j}$ with j even, and by the previous stuff on the total number of subsets, we see it suffices to prove that there are exactly 2^{n-1} subsets with an even number of elements. These subsets come in two varieties, those which contain n and those that don't. The former variety are built from a subset of $\{1, 2, \ldots, n-1\}$ with an odd number of elements, and the latter variety are themselves subsets of $\{1, 2, \ldots, n-1\}$ with an even number of elements. So, there is a 1:1 correspondence of subsets of $\{1, 2, \ldots, n\}$ with an even number of elements and all subsets of $\{1, 2, \ldots, n-1\}$. The latter count is 2^{n-1}, so the former count is as well.

   Solution 3: We use Pascal's formula, so that

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k = \sum_{k=0}^{n} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) (-1)^k,$$

   where we understand $\binom{n-1}{-1}$ and $\binom{n-1}{n}$ to be 0. All of the terms in this sum involve numbers $\binom{n-1}{j}$, so lets see what a particular one contributes to the full sum. It appears with coefficient $(-1)^{j+1}$ when $j = k-1$ and it appears with coefficient $(-1)^j$ when $j = k$. So, in all, $\binom{n-1}{j}$ appears with coefficient $(-1)^{j+1} + (-1)^j = 0$. Thus, the whole sum collapses to 0.

6. True or False, and explain why: A random subset of a 10-element set has between 3 and 7 elements, inclusive, with probability at least 0.9.

   The sample space has size $2^{10} = 1024$. The complementary event has size the sum of $\binom{10}{j}$ for $j = 0, 1, 2, 8, 9, 10$. The latter 3 are equal to the former 3, so the size of the complementary event is $2(1 + 10 + 45) = 112$. Thus, the complementary probability is $112/1024 > 1/10$, so the assertion is False.

7. A subset of size 4 is chosen from $\{1, 2, \dots, 40\}$. True or false, and explain: The probability of the subset having exactly 1 number from each decade (that is, 1 number from 1 to 10, 1 number from 11 to 20, etc.), is at least 0.1.

   The sample space has size $\binom{40}{4} = (40 \cdot 39 \cdot 38 \cdot 37)/24$. The size of the event is 10^4 (there are 10 choices from each decade). Thus, the probability is

$$\frac{10^4 \cdot 24}{40 \cdot 39 \cdot 38 \cdot 37} = \frac{10^3}{13 \cdot 19 \cdot 37}.$$

   Biting the bullet and doing some arithmetic, the last denominator is 9139, so the probability is $> 1/10$ and the assertion is True.

8. What is the conditional probability that a fair coin comes up heads on the fourth flip if it came up heads on the first 3 flips?

   Since the coin is fair, the 4th flip is independent of the first 3 flips, so the probability is 1/2.

9. Prove that if events A, B on a sample space Ω are independent, then so too are events $A, \tilde{B}$.

   There are several equivalent conditions for events C, D to be independent, and one of them is $P(C \cap D) = P(C)P(D)$. So, we are given that $P(A \cap B) = P(A)P(B)$ and we wish to show that $P(A \cap \tilde{B}) = P(A)P(\tilde{B})$. The set A is the disjoint union of $A \cap B$ and $A \cap \tilde{B}$, so $P(A) = P(A \cap B) + P(A \cap \tilde{B})$. Solving this equation for the last term and using that A, B are independent, we have

$$P(A \cap \tilde{B}) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(\tilde{B}),$$

   which proves the assertion. More or less equivalent calculations work using conditional probability.

10. A medical test for a disease gives false positives 10% of the time and false negatives 10% of the time. The chance a random person has the disease is also 10%. If you have the test done and it's positive, what is the probability that you have the disease?

    Let D be the event that you have the disease and T the event that you test positive. We're given:

$$P(D) = .1, \quad P(\tilde{T}|D) = .1, \quad P(T|\tilde{D}) = .1,$$

    and we're being asked for $P(D|T)$. By Bayes, this is

$$\frac{P(D \cap T)}{P(D \cap T) + P(D \cap \tilde{T})}.$$

    The given info can be used to find

$$P(T \cap D) = P(T|D)P(D) = (1 - P(\tilde{T}|D))P(D) = .09$$

and

$$P(T \cap \tilde{D}) = P(T|\tilde{D})P(\tilde{D}) = P(T|\tilde{D})(1 - P(D)) = .09.$$

Thus, the probability we're seeking is $.09/(.09 + .09) = 1/2$.