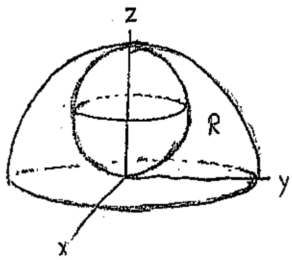
1. (25) Find the centroid of the region R consisting of the solid hemisphere $x^2+y^2+z^2 \le 1$, $z \ge 0$ with the region inside the smaller sphere $x^2+y^2+z^2 = z$ removed. You may use the fact that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.



Small sphere: $x^2+y^2+z^2=z$, $x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$

center $(0,0,\frac{1}{2})$

radius $\frac{1}{2}$

In spherical coordinates:

$\rho^2 = \rho\cos\varphi \longrightarrow \rho = \cos\varphi$

$R: \cos\varphi \le \rho \le 1, \ 0 \le \varphi \le \pi/2, \ 0 \le \theta \le 2\pi$

$\bar{x} = \bar{y} = 0$ by symmetry

Volume of $R = \frac{4}{3}\pi(\frac{1}{2}) - \frac{4}{3}\pi(\frac{1}{8}) = \frac{\pi}{2}$

$$\bar{z} = \frac{\iiint_R z\,dV}{\text{Vol}(R)} = \frac{2}{\pi}\iiint_R z\,dV$$

$$= \frac{2}{\pi}\int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\varphi}^{1} \rho\cos\varphi\,\rho^2\sin\varphi\,d\varphi$$

$$= 4\int_0^{\pi/2} \cos\varphi\sin\varphi \left.\frac{\rho^4}{4}\right|_{\cos\varphi}^{1} d\varphi$$

$$= \int_0^{\pi/2}(1-\cos^4\varphi)\cos\varphi\sin\varphi\,d\varphi = -\int_0^1 u(1-u^4)\,u\,du$$

let $u = \cos\varphi$

$du = -\sin\varphi\,d\varphi$

$$= \frac{u^2}{2} - \frac{u^6}{6}\Big|_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

$\boxed{\text{centroid is } (0,0,\frac{1}{3})}$

2. (15) Consider the function $T(u,v) = (u^2 - v^2, 2uv)$. Then T transforms the rectangle given by $1 \le u \le 2$, $1 \le v \le 3$ into a region R in the xy-plane. Find the area of R.

$$\text{Area}(R) = \iint_R dx\,dy = \iint_{R^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$ by change of variables

R^* is the region in the uv-plane $1 \le u \le 2$, $1 \le v \le 3$

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so, $$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 - (-4v^2) = 4u^2 + 4v^2.$$

$$\iint_R dx\,dy = \int_1^3 \int_1^2 (4u^2 + 4v^2)\,du\,dv$$

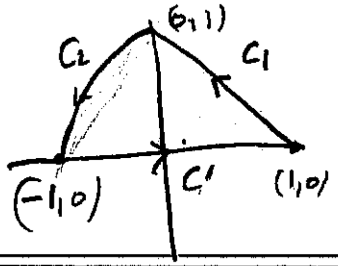
$$= \int_1^3 \left(\frac{4u^3}{3} + 4uv^2 \right) \Big|_1^2 dv$$

$$= \int_1^3 \left(\frac{28}{3} + 4v^2 \right) dv$$

$$= \frac{28}{3}v + \frac{4v^3}{3} \Big|_1^3 = \left(\frac{84}{3} + 36 \right) - \left(\frac{28}{3} + \frac{4}{3} \right)$$

$$= \frac{52}{3} + 36 = \boxed{\frac{160}{3}}$$

3. (25) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F}(x,y) = (xy, \sin(y^7))$ and C is the oriented curve consisting of the straight line segment from (1,0) to (0,1) followed by the portion of the unit circle from (0,1) to (-1,0).



$C = C_1 \cup C_2$ C' oriented line segment from $(-1,0)$ to $(1,0)$

Apply G.T. to region D enclosed by $C \cup C'$

$$P = xy, \quad Q = \sin(y^7), \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -x$$

$$\therefore \int_{C\cup C'} F = -\iint_D x\, dA = -\int_0^1 dy \int_{-\sqrt{1-y^2}}^{1-y} x\, dx =$$

$$-\frac{1}{2}\int_0^1 dy\; x^2\Big]_{-\sqrt{1-y^2}}^{1-y} = \frac{1}{2}\int_0^1 \left((1-y)^2 - (1-y^2)\right) dy =$$

$$-\frac{1}{2}\int_0^1 (2y^2 - 2y)\, dy = -\int_0^1 (y^2 - y)\, dy = -\left(\frac{y^3}{3} - \frac{y^2}{2}\right)\Big|_0^1 = -\frac{1}{3} + \frac{1}{2} = +\frac{1}{6}$$

$\int_{C'} xy\,dx + \sin y^7\, dy = 0$ since $y = 0$ on C'

$$\therefore \int_{C\cup C'} F = \int_C F + \int_{C'} F = \int_C F$$

$\|$ G.T

$\frac{1}{6}$

Ans $\frac{1}{6}$

4. (15) Let C be any curve in the xy-plane from (1,1) to (2,2) which does not pass through the origin and which is oriented from (1,1) to (2,2) and let $\mathbf{F}$ be the vector field given by

$$\mathbf{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

Evaluate $\int_C \mathbf{F} \cdot ds$.

Notice that $\bar{F}$ is conservative

$$\nabla \times \bar{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/x^2+y^2 & y/x^2+y^2 & 0 \end{vmatrix} = \langle 0,0,0 \rangle.$$

so $\bar{F} = \nabla f$ for some function f.

$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}$ $\quad f(x,y) = \int \frac{x}{x^2+y^2}\,dx = \frac{1}{2}\ln(x^2+y^2) + g(y)$

$\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$ $\quad$ ↑ from above we have $\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2} + g'(y)$

so $g'(y)=0$ and hence $g(y)=C$.

choose $g(y)=0$.

So, we have $\bar{F} = \nabla f$ where $f(x,y) = \frac{1}{2}\ln(x^2+y^2)$.

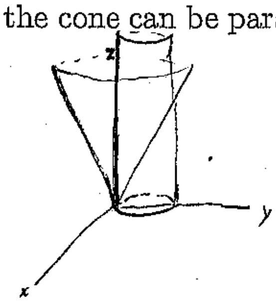
$$\int_C \bar{F} \cdot d\bar{s} = f(2,2) - f(1,1)$$
$$= \frac{1}{2}\ln(8) - \frac{1}{2}\ln(2)$$
$$= \frac{3}{2}\ln(2) - \frac{1}{2}\ln(2) = \ln(2).$$

$$\boxed{\int_C \bar{F} \cdot d\bar{s} = \ln(2)}$$

5. (20) Find the surface area of the portion of the cone $z = \sqrt{4x^2 + 4y^2}$ which is inside the cylinder $x^2 + y^2 = 4y$. Express your answer as an iterated integral with limits of integration included. Do not evaluate the integral. Hint: the cone can be parametrized by $x = r\cos\theta$, $y = r\sin\theta$, $z = 2r$.

$r^2 = 4r\sin\theta$

or $r = 4\sin\theta$.



With $\vec{X}(r,\theta) = (r\cos\theta, r\sin\theta, 2r)$, $0 \le \theta \le \pi$, $0 \le r \le 4\sin\theta$;

$\vec{T}_r = (\cos\theta, \sin\theta, 2)$,

$\vec{T}_\theta = (-r\sin\theta, r\cos\theta, 0)$ and $\|\vec{T}_r \times \vec{T}_\theta\| = \|(-2r\cos\theta, 2r\sin\theta, r)\|$

$= \sqrt{4r^2(\cos^2\theta + \sin^2\theta) + r^2}$

$= \sqrt{5}\,r$.

So surface area $= \iint dS = \iint \|\vec{T}_r \times \vec{T}_\theta\|\, dr\, d\theta$

$= \int_{\theta=0}^{\pi}\int_{r=0}^{4\sin\theta} \sqrt{5}\, r\, dr\, d\theta$

$= \sqrt{5}\int_0^{\pi}\int_0^{4\sin\theta} r\, dr\, d\theta$

$= \sqrt{5}$ (area of circle $r = 4\sin\theta$)

$= \sqrt{5}\,\pi\, 2^2$