

Math 123 Homework Assignment #4
Due at the end of term (optional)

Part I:

1. Let $\{A_n, \varphi_n\}$ be a direct sequence of C^* -algebras in which all the connecting maps φ_n are injective. Let (A, φ^n) be “the” direct limit.

(a) Show that $\varphi^n(a) = \varphi^m(b)$ if and only if there is a $k \geq \max n, m$ such that $\varphi_{n,k}(a) = \varphi_{m,k}(b)$, and show that each φ^n is injective.

2. Let A be a C^* -algebra. We call $u \in A$ a *partial isometry* if u^*u is a projection. Show that the following are equivalent.

(a) u is a partial isometry.

(b) $u = uu^*u$.

(c) $u^* = u^*uu^*$.

(d) uu^* is a projection.

(e) u^* is a partial isometry.

(Suggestion: for (b) \implies (a), use the C^* -norm identity on $\|uu^*u - u\| = \|u(u^*u - 1)\|$.)

Part II:

3. Give the details of the argument sketched in lecture that if A is a finite-dimensional C^* -algebra, then $A \cong \bigoplus_{i=1}^n M_{n_i}$. (Recall that you can use the argument of Corollary ??¹ to conclude that A has a finite-dimensional faithful representation. Then we may as well assume that $A \subset \mathcal{K}(\mathcal{H}) = B(\mathcal{H})$. Now apply Theorem AU to the identity representation of A .)

4. Let $T = (T_{ij})$ be an operator in $M_n(B(\mathcal{H}))$. Show that

$$\|T_{ij}\| \leq \|(T_{ij})\| \leq \sum_{ij} \|T_{ij}\|.$$

¹Sorry, I have lost track of the numbers. I tried.

5. Let M_s be a UHF algebra such which is not a matrix algebra. (This is automatic if $s : \mathbb{Z}^+ \rightarrow \{2, 3, \dots\}$.) Show that M_s is not GCR.

6. Recall the following notations. Let $\bar{\mathbf{n}} = (n_1, \dots, n_k) \in (\mathbb{Z}^+)^k$, and $|\bar{\mathbf{n}}| = n_1 + \dots + n_k$. View elements of $M_{\bar{\mathbf{n}}} = M_{n_1} \oplus \dots \oplus M_{n_k}$ as block diagonal matrices in $M_{|\bar{\mathbf{n}}|}$. Recall that $M = (m_{ij}) \in M_{s \times k}(\mathbb{N})$ is called admissible if $\sum_{j=1}^k m_{ij} n_j \leq r_i$ for all $i = 1, 2, \dots, s$. We defined $\varphi_M : M_{\bar{\mathbf{n}}} \rightarrow M_{\bar{\mathbf{r}}}$ by $\varphi_M(T_1 \oplus \dots \oplus T_k) = (T'_1 \oplus \dots \oplus T'_s)$ where

$$T'_i = m_{i1} \cdot T_1 \oplus \dots \oplus m_{ik} \cdot T_k \oplus \mathbf{0}_{d_i},$$

with $d_i = r_i - \sum_{j=1}^k m_{ij} n_j$, and $m \cdot T = \underbrace{T \oplus \dots \oplus T}_{m \text{ times}}$.

Prove that if $\varphi : M_{\bar{\mathbf{n}}} \rightarrow M_{\bar{\mathbf{r}}}$ is a $*$ -homomorphism, then there is a unitary $u \in M_{\bar{\mathbf{r}}}$ such that $\varphi = \text{Ad } u \circ \varphi_M$ for some admissible matrix M .

(Suggestions: (1) View φ as a (possibly degenerate) representation of $M_{\bar{\mathbf{n}}} \subseteq M_{|\bar{\mathbf{n}}|} = B(\mathbb{C}^{|\bar{\mathbf{n}}|})$ into $M_{\bar{\mathbf{r}}} \subseteq B(\mathbb{C}^{|\bar{\mathbf{r}}|})$. Use Theorem AP to write φ as $\sum_i \pi^i$, where each π^i is an irreducible subrepresentation equivalent to a subrepresentation of $\text{id} : M_{\bar{\mathbf{n}}} \rightarrow B(\mathbb{C}^{|\bar{\mathbf{n}}|})$. (2) Conclude that $\varphi = \sum_{i=1}^s \varphi_i$ where each φ_i is a $*$ -homomorphism of $M_{\bar{\mathbf{n}}}$ into $M_{r_i} = B(\mathbb{C}^{r_i})$. Then use Theorem AP again to see that $\varphi_i \cong \bigoplus_{j=1}^k m_{ij} \cdot \text{id}_{M_{n_j}}$. (3) Now show that there is a $U \in M_{r_j}$ such that $\varphi_i(T_1 \oplus \dots \oplus T_k) = U(m_{i1} \cdot T_1 \oplus \dots \oplus m_{ik} \cdot T_k \oplus \mathbf{0}_{d_i})U^*$.)

7. Suppose that $(A, \{\varphi^n\})$ is the direct limit of a direct system $\{(A_n, \varphi_n)\}$, and that $(B, \{\psi^n\})$ is the direct limit of another direct system $\{(B_n, \psi_n)\}$. Suppose that there are maps $\alpha^n : A_n \rightarrow B_n$ such that the diagrams

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ \alpha^n \downarrow & & \downarrow \alpha^{n+1} \\ B_n & \xrightarrow{\psi_n} & B_{n+1} \end{array}$$

commute for all n . Show that there is a unique homomorphism $\alpha : A \rightarrow B$ such that

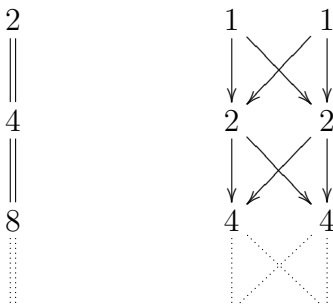
$$\begin{array}{ccc} A_n & \xrightarrow{\alpha^n} & B_n \\ \varphi^n \downarrow & & \downarrow \psi^n \\ A & \xrightarrow{\alpha} & B \end{array}$$

commutes for all n .

8. Let J be an ideal in a C^* -algebra A . We call J a *primitive* ideal if $J = \ker \pi$ for some irreducible representation π of A . On the other hand, J is called *prime* if whenever I_1 and I_2 are ideal in A such that $I_1 I_2 \subseteq J$, then either $I_1 \subseteq J$ or $I_2 \subseteq J$. Show that every primitive ideal in a C^* -algebra is prime². (Suggestion: If $I \not\subseteq J$, then $\pi|_I$ is irreducible and $[\pi(I)\xi] = \mathcal{H}$ for all $\xi \in \mathcal{H}$.)

Part III:

9. Let A be the UHF algebra M_s where $s = (2, 2, 2, \dots)$. Thus A is the C^* -direct limit arising from the maps $\varphi_M : M_{2^n} \rightarrow M_{2^{n+1}}$ with $M = (2)$. Let B be the C^* -direct limit arising from the maps $\varphi_{M'} : M_{(2^{n-1}, 2^{n-1})} \rightarrow M_{(2^n, 2^n)}$ with $M' = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. In terms of Bratteli diagrams



Use Elliot’s Theorem to show that A and B are isomorphic.

10. Let A be a C^* -algebra without unit. Then \tilde{A} is the smallest C^* -algebra with unit containing A as an ideal. It has become apparent in the last few years, that it is convenient to work with the “largest” such algebra (in a sense to be made precise below). For motivation, suppose that A sits in B as an ideal. Then each $b \in B$ defines a pair of operators $L, R \in B(A)$ defined by $L(a) = ba$ and $R(a) = ab$. Note that for all $a, c \in A$,

$$(1) \quad L(ac) = L(a)c, \quad (2) \quad R(ac) = aR(c), \quad (3) \quad aL(c) = R(a)c.$$

Define a *multiplier* or *double centralizer* on A to be a pair (L, R) of operators on A satisfying conditions (1), (2), and (3) above. Let $\mathcal{M}(A)$ denote the set of all multipliers on A .

- (a) If $(L, R) \in \mathcal{M}(A)$, then use the closed graph theorem to show that L and R must be bounded, and that $\|L\| = \|R\|$.

²If A is separable, the converse holds. It has just recently been discovered that the converse can fail without the separable assumption.

- (b) Define operations and a norm on $\mathcal{M}(A)$ so that $\mathcal{M}(A)$ becomes a C^* -algebra which contains A as an ideal. (Use the example of A sitting in B as an ideal above for motivation.)
- (c) An ideal A in B is called *essential* if the only ideal J in B such that $AJ = \{0\}$ is $J = \{0\}$. Show that A is an essential ideal in $\mathcal{M}(A)$. Also show that if A is an essential ideal in a B , then there is an injective $*$ -homomorphism of B into $\mathcal{M}(A)$ which is the identity on A .
- (d) Compute $\mathcal{M}(A)$ for $A = C_0(X)$ and $A = \mathcal{K}(\mathcal{H})$.

11. Let $\text{Prim}(A)$ be the set of primitive ideals of a C^* -algebra A . If $S \subseteq \text{Prim}(A)$, then define $\ker(S) = \bigcap_{P \in S} P$ (with $\ker(\emptyset) = A$). Also if I is an ideal in A , then define $\text{hull}(I) = \{P \in \text{Prim}(A) : I \subseteq P\}$. Finally, for each $S \in \text{Prim}(A)$, set $\overline{S} = \text{hull}(\ker(S))$.

- (a) Show that if $R_1, R_2 \subset \text{Prim}(A)$, then $\overline{R_1 \cup R_2} = \overline{R_1} \cup \overline{R_2}$.
- (b) Show that if $R_\lambda \in \text{Prim}(A)$ for all $\lambda \in \Lambda$, then $\overline{\bigcap_{\lambda \in \Lambda} R_\lambda} = \bigcap_{\lambda \in \Lambda} \overline{R_\lambda}$.
- (c) Conclude that there is a unique topology on $\text{Prim}(A)$ so that $\{\overline{S} : S \subseteq \text{Prim}(A)\}$ are the closed subsets.

This topology is called the *Hull-Kernel* or *Jacobson* topology.

12. Consider the C^* -algebras

- (a) $A = C_0(X)$, with X locally compact Hausdorff.
- (b) $B = C([0, 1], M_2)$, the set of continuous functions from $[0, 1]$ to M_2 with the sup-norm and pointwise operations.
- (c) $C = \{f \in B : f(0) = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \alpha \in \mathbb{C}\}$.
- (d) $D = \{f \in B : f(0) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \alpha, \beta \in \mathbb{C}\}$.

For each of the above discuss the primitive ideal space and its topology. For example, show that $\text{Prim}(A)$ is homeomorphic to X . Notice that all of the above are CCR.