Math 123 Homework Assignment #4
Due at the end of term (optional)

Part I:

1. Let $\{ A_n, \varphi_n \}$ be a direct sequence of C^*-algebras in which all the connecting maps φ_n are injective. Let (A, φ^n) be "the" direct limit.

(a) Show that $\varphi^n(a) = \varphi^m(b)$ if and only if there is a $k \geq \max n, m$ such that $\varphi_{n,k}(a) = \varphi_{m,k}(b)$, and

(b) Show that each φ^n is injective.

2. Let A be a C^*-algebra. We call $u \in A$ a *partial isometry* if u^*u is a projection. Show that the following are equivalent.

(a) u is a partial isometry.

(b) $u = uu^*u$.

(c) $u^* = u^*uu^*$.

(d) uu^* is a projection.

(e) u^* is a partial isometry.

(Suggestion: for (b) $\Longrightarrow$ (a), use the C^*-norm identity on $\|uu^*u - u\| = \|u(u^*u - 1)\|$.)

Part II:

3. Give the details of the argument sketched in lecture that if A is a finite-dimensional C^*-algebra, then $A \cong \bigoplus_{i=1}^n M_{n_i}$. (Recall that you can use the argument of Corollary BG to conclude that A has a finite-dimensional faithful representation. Then we may as well assume that $A \subset \mathcal{K}(\mathcal{H}) = B(\mathcal{H})$. Now apply Theorem AP to the identity representation of A.)

4. Let $T = (T_{ij})$ be an operator in $M_n\big(B(\mathcal{H})\big)$. Show that

$$\|T_{ij}\| \leq \big\|(T_{ij})\big\| \leq \sum_{ij} \|T_{ij}\|.$$

5. Let M_s be a UHF algebra such which is not a matrix algebra. (This is automatic if $s : \mathbb{Z}^+ \to \{ 2, 3, \dots \}$.) Show that M_s is not GCR.

6. Recall the following notations. Let $\overline{\mathbf{n}} = (n_1, \dots, n_k) \in (\mathbb{Z}^+)^k$, and $|\overline{\mathbf{n}}| = n_1 + \cdots n_k$. View elements of $M_{\overline{\mathbf{n}}} = M_{n_1} \oplus \cdots \oplus M_{n_k}$ as block diagonal matrices in $M_{|\overline{\mathbf{n}}|}$. Recall that $M = (m_{ij}) \in M_{s\times k}(\mathbb{N})$ is called admissible if $\sum_{j=1}^k m_{ij} n_j \le r_i$ for all $i = 1, 2, \dots, s$. We defined $\varphi_M : M_{\overline{\mathbf{n}}} \to M_{\overline{\mathbf{r}}}$ by $\varphi_M(T_1 \oplus \cdots \oplus T_k) = (T_1' \oplus \cdots \oplus T_s')$ where

$$T_i' = m_{i1} \cdot T_1 \oplus \cdots \oplus m_{ik} \cdot T_k \oplus \mathbf{0}_{d_i},$$

with $d_i = r_i - \sum_{j=1}^k m_{ij} n_j$, and $m \cdot T = \underbrace{T \oplus \cdots \oplus T}_{m \text{ times}}$.

Prove that if $\varphi : M_{\overline{\mathbf{n}}} \to M_{\overline{\mathbf{r}}}$ is a $*$-homomorphism, then there is a unitary $u \in M_{\overline{\mathbf{r}}}$ such that $\varphi = \operatorname{Ad} u \circ \varphi_M$ for some admissible matrix M.
(Suggestions: (1) View φ as a (possibly degenerate) representation of $M_{\overline{\mathbf{n}}} \subseteq M_{|\overline{\mathbf{n}}|} = B(\mathbb{C}^{|\overline{\mathbf{n}}|})$ into $M_{\overline{\mathbf{r}}} \subseteq B(\mathbb{C}^{|\overline{\mathbf{r}}|})$. Use Theorem AP to write φ as $\sum_i \pi^i$, where each π^i is an irreducible subrepresentation equivalent to a subrepresentation of $\mathrm{id} : M_{\overline{\mathbf{n}}} \to B(\mathbb{C}^{|\overline{\mathbf{n}}|})$. (2) Conclude that $\varphi = \sum_{i=1}^s \varphi_i$ where each φ_i is a $*$-homomorphism of $M_{\overline{\mathbf{n}}}$ into $M_{r_i} = B(\mathbb{C}^{r_i})$. Then use Theorem AP again to see that $\varphi_i \cong \bigoplus_{j=1}^k m_{ij} \cdot \mathrm{id}_{M_{n_j}}$. (3) Now show that there is a $U \in M_{r_j}$ such that $\varphi_i(T_1 \oplus \cdots \oplus T_k) = U(m_{i1} \cdot T_1 \oplus \cdots \oplus m_{ik} \cdot T_k \oplus \mathbf{0}_{d_i})U^*$.)

7. Suppose that $\big(A, \{ \varphi^n \}\big)$ is the direct limit of a direct system $\{ (A_n, \varphi_n) \}$, and that $\big(B, \{ \psi^n \}\big)$ is the direct limit of another direct system $\{ (B_n, \psi_n) \}$. Suppose that there are maps $\alpha^n : A_n \to B_n$ such that the diagrams

$$\begin{tikzcd} A_n \arrow[r, "\varphi_n"] \arrow[d, "\alpha^n"'] & A_{n+1} \arrow[d, "\alpha^{n+1}"] \\ B_n \arrow[r, "\psi_n"'] & B_{n+1} \end{tikzcd}$$

commute for all n. Show that there is a unique homomorphism $\alpha : A \to B$ such that

$$\begin{tikzcd} A_n \arrow[r, "\alpha^n"] \arrow[d, "\varphi^n"'] & B_n \arrow[d, "\psi^n"] \\ A \arrow[r, "\alpha"'] & B \end{tikzcd}$$

commutes for all n.

8. Let J be an ideal in a C*-algebra A. We call J a *primitive* ideal if $J = \ker \pi$ for some irreducible representation π of A. On the other hand, J is called *prime* if whenever I_1 and I_2 are ideal in A such that $I_1 I_2 \subseteq J$, then either $I_1 \in J$ or $I_2 \in J$. Show that every primitive ideal in a C*-algebra is prime[1]. (Suggestion: If $I \not\subseteq J$, then $\pi|_I$ is irreducible and $[\pi(I)\xi] = \mathcal{H}$ for all $\xi \in \mathcal{H}$.)

Part III:

9. Let A be the UHF algebra M_s where $s = (2, 2, 2, \ldots)$. Thus A is the C^*-direct limit arising from the maps $\varphi_M : M_{2^n} \to M_{2^{n+1}}$ with $M = (2)$. Let B be the C^*-direct limit arising from the maps $\varphi_{M'} : M_{(2^{n-1}, 2^{n-1})} \to M_{(2^n, 2^n)}$ with $M' = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. In terms of Bratteli diagrams

$$\begin{tikzcd} 2 \arrow[d, equal] & & 1 \arrow[d] \arrow[dr] & 1 \arrow[d] \arrow[dl] \\ 4 \arrow[d, equal] & & 2 \arrow[d] \arrow[dr] & 2 \arrow[d] \arrow[dl] \\ 8 \arrow[d, dotted, no head] & & 4 \arrow[d, dotted, no head] \arrow[dr, dotted, no head] & 4 \arrow[d, dotted, no head] \arrow[dl, dotted, no head] \\ {} & & {} & {} \end{tikzcd}$$

Use Elliot's Theorem to show that A and B are isomorphic.

10. Let A be a C^*-algebra without unit. Then $\widetilde{A}$ is the smallest C^*-algebra with unit containing A as an ideal. It has become apparent in the last few years, that it is convenient to work with the "largest" such algebra (in a sense to be made precise below). For motivation, suppose that A sits in B as an ideal. Then each $b \in B$ defines a pair of operators $L, R \in B(A)$ defined by $L(a) = ba$ and $R(a) = ab$. Note that for all $a, c \in A$,

$$(1) \quad L(ac) = L(a)c, \qquad (2) \quad R(ac) = aR(c), \qquad (3) \quad aL(c) = R(a)c.$$

Define a *multiplier* or *double centralizer* on A to be a pair (L, R) of operators on A satisfying conditions (1), (2), and (3) above. Let $\mathcal{M}(A)$ denote the set of all multipliers on A.

(a) If $(L, R) \in \mathcal{M}(A)$, then use the closed graph theorem to show that L and R must be bounded, and that $\|L\| = \|R\|$.

[1]If A is separable, the converse holds. It has just recently been discovered that the converse can fail without the separable assumption.

(b) Define operations and a norm on $\mathcal{M}(A)$ so that $\mathcal{M}(A)$ becomes a C^*-algebra which contains A as an ideal. (Use the example of A sitting in B as an ideal above for motivation.)

(c) An ideal A in B is called *essential* if the only ideal J in B such that $AJ = \{0\}$ is $J = \{0\}$. Show that A is an essential ideal in $\mathcal{M}(A)$. Also show that if A is an essential ideal in a B, then there is an injective $*$-homomorphism of B into $\mathcal{M}(A)$ which is the identity on A.

(d) Compute $\mathcal{M}(A)$ for $A = C_0(X)$ and $A = \mathcal{K}(\mathcal{H})$.

11. Let $\operatorname{Prim}(A)$ be the set of primitive ideals of a C*-algebra A. If $S \subseteq \operatorname{Prim}(A)$, then define $\ker(S) = \bigcap_{P\in S} P$ (with $\ker(\emptyset) = A$). Also if I is an ideal in A, then define $\operatorname{hull}(I) = \{ P \in \operatorname{Prim}(A) : I \subseteq P \}$. Finally, for each $S \in \operatorname{Prim}(A)$, set $\overline{S} = \operatorname{hull}\bigl(\ker(S)\bigr)$.

(a) Show that if $R_1, R_2 \subset \operatorname{Prim}(A)$, then $\overline{R_1 \cup R_2} = \overline{R_1} \cup \overline{R_2}$.

(b) Show that if $R_\lambda \in \operatorname{Prim}(A)$ for all $\lambda \in \Lambda$, then $\overline{\bigcap_{\lambda\in\Lambda} \overline{R_\lambda}} = \bigcap_{\lambda\in\Lambda} \overline{R_\lambda}$.

(c) Conclude that there is a unique topology on $\operatorname{Prim}(A)$ so that $\{ \overline{S} : S \subseteq \operatorname{Prim}(A) \}$ are the closed subsets.

This topology is called the *Hull-Kernel* or *Jacobson* topology.

12. Consider the C*-algebras

(a) $A = C_0(X)$, with X locally compact Hausdorff.

(b) $B = C\bigl([0,1], M_2\bigr)$, the set of continuous functions from $[0,1]$ to M_2 with the sup-norm and pointwise operations.

(c) $C = \{ f \in B : f(0) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right), \alpha \in \mathbb{C} \}$.

(d) $D = \{ f \in B : f(0) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right), \alpha, \beta \in \mathbb{C} \}$.

For each of the above discuss the primitive ideal space and its topology. For example, show that $\operatorname{Prim}(A)$ is homeomorphic to X. Notice that all of the above are CCR.