

Math 123 Homework Assignment #1

Due Friday, April 11th

Part I:

1. Let X be a normed vector space and suppose that S and T are bounded linear operators on X . Show that $\|ST\| \leq \|S\|\|T\|$.
2. Let X be a locally compact Hausdorff space. Show that $C_0(X)$ is a closed subalgebra of $C^b(X)$.
3. Let A be a unital Banach algebra. Show that $x \mapsto x^{-1}$ is continuous from $G(A)$ to $G(A)$. (Use the hint given in lecture.)

ANS: Using the “hint”, suppose that $a \in G(A)$ and let $b = a - h$ for some $h \in A$ with

$$\|h\| \leq \frac{\|a^{-1}\|^{-1}}{2}. \quad (1)$$

Then standard arguments show that $b \in G(A)$. (Recall that if $\|c\| < 1$, then $1 - c \in G(A)$ and $(1 - c)^{-1} = 1 + c + c^2 + \dots$.) Now we observe that

$$\begin{aligned} \|b^{-1} - a^{-1}\| &= \|(a - h)^{-1} - a^{-1}\| \\ &= \|((1 - a^{-1}h) - 1)a^{-1}\| \\ &\leq \left\| \sum_{n=1}^{\infty} (a^{-1}h)^n \right\| \|a^{-1}\| \\ &\leq \sum_{n=1}^{\infty} \|a^{-1}h\|^n \|a^{-1}\| \\ &= \frac{\|a^{-1}h\| \|a^{-1}\|}{1 - \|a^{-1}h\|} \end{aligned}$$

which, in view of (1) and $\|a^{-1}h\| \leq \|a^{-1}\|\|h\|$, is

$$\leq 2\|a^{-1}\|^2\|h\|.$$

Since the latter term tends to 0 with h , we're done.

However, Peng Peng Yu came up with a cleaner argument that does not require A to be a Banach algebra. Here it is enough that A be a normed algebra (and therefore with a submultiplicative norm).

Fix $a \in A$ and $\epsilon > 0$. Let $\delta = \frac{1}{2} \min(\epsilon \|a^{-1}\|^{-2}, \|a^{-1}\|^{-1})$. We just need to verify that if $b \in B_\delta(a) \cap G(A)$, then $\|a^{-1} - b^{-1}\| < \epsilon$. However, we first notice that by the “reverse triangle inequality”,

$$\begin{aligned} \|b^{-1}\| - \|a^{-1}\| &\leq \|b^{-1} - a^{-1}\| \\ &= \|b^{-1}(a - b)a^{-1}\| \\ &\leq \|b^{-1}\| \|a - b\| \|a^{-1}\| \end{aligned}$$

which, since $\|a - b\| < \frac{1}{2} \|a^{-1}\|^{-1}$, is

$$< \frac{1}{2} \|b^{-1}\|.$$

In particular, this implies that

$$\|b^{-1}\| < 2\|a^{-1}\| \quad \text{if } b \in B_\delta(a) \cap G(A). \quad (2)$$

But then if $b \in B_\delta(a) \cap G(A)$, we have (2) and $\|a - b\| < \frac{\epsilon}{2} \|a^{-1}\|^{-2}$. Therefore

$$\begin{aligned} \|b^{-1} - a^{-1}\| &= \|b^{-1}(a - b)a^{-1}\| \\ &\leq \|b^{-1}\| \|a - b\| \|a^{-1}\| \\ &< \epsilon. \end{aligned}$$

This is what we wanted to show.

Part II:

4. Suppose that X is a compact Hausdorff space. If E is a closed subset of X , define $I(E)$ to be the ideal in $C(X)$ of functions which vanish on E .

- (a) Let J be a closed ideal in $C(X)$ and let $E = \{x \in X : f(x) = 0 \text{ for all } f \in J\}$. Prove that if U is an open neighborhood of E in X , then there is a $f \in J$ such that $f(x) = 1$ for all x in the compact set $X \setminus U$.
- (b) Conclude that $J = I(E)$ in part (a), and hence, conclude that *every closed* ideal in $C(X)$ has the form $I(E)$ for some closed subset E of X .

ANS: Fix $x_0 \in X \setminus U$. By definition of E , there is a $f_{x_0} \in J$ with $f_{x_0}(x_0) \neq 0$. Since $|f|^2 = \bar{f}f \in J$ if $f \in J$, we may as well assume that $f_{x_0}(x) \geq 0$ for all $x \in X$, and since J is a subalgebra, we may also assume that $f_{x_0}(x_0) > 1$. Since $X \setminus U$ is compact, there are $x_1, \dots, x_n \in X$ so that $f = \sum_k f_{x_k}$ satisfies $f \in J$ and $f(x) > 1$ for all $x \in X \setminus U$. Observe that $g = \min(1, 1/f)$ is in $C(X)$ ¹. Since $fg \in J$, we are done with part (a).

¹If $a, b \in C(X)$, then so are $\min(a, b) = (a+b)/2 - |a-b|/2$ and $\max(a, b) = (a+b)/2 + |a-b|/2$. In the above, we can replace f by $\max(f, 1/2)$ without altering g .

Notice that we have proved a bit more than required in part (a): namely there is a $f \in J$ such that $0 \leq f(x) \leq 1$ for all $x \in X$ and $f(x) = 1$ for all $x \notin U$. Thus if h is any function in $I(E)$ and $\epsilon > 0$, then $U = \{x \in X : |h(x)| > \epsilon\}$ is a neighborhood of E in X . Then we can choose $f \in J$ as above and $\|fh - h\|_\infty < \epsilon$. Thus $h \in \overline{J} = J$. This suffices as we have $J \subseteq I(E)$ by definition. (Notice that if $E = \emptyset$, then we can take $U = \emptyset$ in the above and then $J = C(X)$.)

Remark: Notice that we have established a 1-1 correspondence between the closed subsets E of X and the closed ideals J of $C(X)$: it follows immediately from Urysohn's Lemma² that if E is closed and $x \notin E$, then there is a $f \in I(E)$ with $f(x) \neq 0$. Thus $I(E) \neq I(F)$ if E and F are distinct closed sets.

5. Suppose that X is a (non-compact) locally compact Hausdorff space. Let X^+ be the *one-point compactification* of X (also called the Alexandroff compactification: see [Kelly; Theorem 5.21]). Recall that $X^+ = X \cup \{\infty\}$ with $U \subseteq X^+$ open if and only if either U is an open subset of X or $X^+ \setminus U$ is a *compact* subset of X .

(a) Show that $f \in C(X)$ belongs to $C_0(X)$ if and only if the extension

$$\tilde{f}(\tilde{x}) = \begin{cases} f(\tilde{x}) & \text{if } \tilde{x} \in X, \text{ and} \\ 0 & \text{if } \tilde{x} = \infty. \end{cases}$$

is continuous on X^+ .

(b) Conclude that $C_0(X)$ can be identified with the maximal ideal of $C(X^+)$ consisting of functions which 'vanish at ∞ .'

ANS: Suppose \tilde{f} is continuous at $x = \infty$, and that $\epsilon > 0$. Then $U = \{\tilde{x} \in X^+ : |\tilde{f}(\tilde{x})| < \epsilon\}$ is an open neighborhood of ∞ in X^+ . But then $X \setminus U$ is compact; but that means $\{x \in X : |f(x)| \geq \epsilon\}$ is compact. That is, $f \in C_0(X)$ as required.

For the converse, suppose that $f \in C_0(X)$, and that V is open in \mathbb{C} . If $0 \notin V$, then $\tilde{f}^{-1}(V) = f^{-1}(V)$ is open in X , and therefore, open in X^+ . On the other hand, if $0 \in V$, then there is a $\epsilon > 0$ so that $\{z \in \mathbb{C} : |z| < \epsilon\} \subseteq V$. Thus, $X^+ \setminus \tilde{f}^{-1}(V) = \{x \in X : f(x) \notin V\} \cap \{x \in X : |f(x)| \geq \epsilon\}$. Since the first set is closed and the second compact, $X^+ \setminus \tilde{f}^{-1}(V)$ is a compact subset of X , and $\tilde{f}^{-1}(V)$ is a open neighborhood of ∞ in X^+ . This proves part (a).

Part (b) is immediate: each $f \in C_0(X)$ has a (unique) extension to a function in $C(X^+)$ and this identifies $C_0(X)$ with the ideal $I(\{\infty\})$ in $C(X^+)$. In view of question 4 above, $I(\{\infty\})$ is maximal among closed ideals in $C(X^+)$, and, as maximal ideals are automatically closed, maximal among all proper ideals.

6. Use the above to establish the following ideal theorem for $C_0(X)$.

²For a reference, see Pedersen's *Analysis Now*: Theorems 1.5.6 and 1.6.6 or, more generally, Proposition 1.7.5.

Theorem: Suppose that X is a locally compact Hausdorff space. Then every closed ideal J in $C_0(X)$ is of the form

$$J = \{ f \in C_0(X) : f(x) = 0 \text{ for all } x \in E \}$$

for some closed subset E of X .

ANS: Suppose that J is a closed ideal in $C_0(X)$. Then J is, in view of question 5(b) above, a closed subalgebra of $C(X^+)$. I claim the result will follow once it is observed that J is actually an ideal in $C(X^+)$. In that case, $J = I(E \cup \{\infty\})$, where $E \subseteq X$ is such that $E \cup \{\infty\}$ is closed in X^+ . Thus $X^+ \setminus (E \cup \{\infty\}) = X \setminus E$ is open in X , and E is closed in X .

The easy way to verify the claim, is to observe that, in view of the fact that $C_0(X)$ is a maximal ideal in $C(X^+)$, $C(X^+) = \{ f + \lambda : f \in C_0(X) \text{ and } \lambda \in \mathbb{C} \}$. (Here $\lambda \in \mathbb{C}$ is identified with the constant function on X^+ .) Then, since J is an algebra, $f(g + \lambda) = fg + \lambda f$ belongs to J whenever f does.

Part III:

7. Assume you remember enough measure theory to show that if $f, g \in L^1([0, 1])$, then

$$f * g(t) = \int_0^t f(t-s)g(s) ds \tag{3}$$

exists for almost all $t \in [0, 1]$, and defines an element of $L^1([0, 1])$. Let A be the algebra consisting of the Banach space $L^1([0, 1])$ with multiplication defined by (3).

- (a) Conclude that A is a commutative Banach algebra: that is, show that $f * g = g * f$, and that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- (b) Let f_0 be the constant function $f_0(t) = 1$ for all $t \in [0, 1]$. Show that

$$f_0^n(t) := f_0 * \cdots * f_0(t) = t^{n-1}/(n-1)!, \tag{4}$$

and hence,

$$\|f_0^n\|_1 = \frac{1}{n!}. \tag{5}$$

- (c) Show that (4) implies that f_0 generates A as a Banach algebra: that is, $\text{alg}(f)$ is norm dense. Conclude from (5) that the spectral radius $\rho(f)$ is zero for all $f \in A$.
- (d) Conclude that A has no nonzero complex homomorphisms.

ANS: First compute that³

$$\begin{aligned}\|f * g\|_1 &= \int_0^1 |f * g(t)| dt \\ &\leq \int_0^1 \int_0^t |f(t-s)g(s)| ds dt\end{aligned}$$

which, using Tonelli's Theorem, is

$$\begin{aligned}&= \int_0^1 |g(s)| \left(\int_s^1 |f(t-s)| dt \right) ds \\ &= \int_0^1 |g(s)| \left(\int_0^{1-s} |f(u)| du \right) ds \\ &\leq \|f\|_1 \|g\|_1.\end{aligned}$$

To show that $f * g = g * f$ it suffices, in view of the above, to consider continuous functions. Thus, the usual calculus techniques apply. In particular,

$$\begin{aligned}f * g(t) &= \int_0^t f(t-s)g(s) ds \\ &= - \int_t^0 f(u)g(t-u) du = g * f(t).\end{aligned}$$

This proves (a). However, (b) is a simple induction argument.

Now for (c): the calculation (4) shows that $\text{alg}(f_0)$ contains all polynomials. Since the polynomials are uniformly dense in $C[0, 1]$, and the latter is dense in L^1 , we can conclude that $\text{alg}(f_0)$ is norm dense.

Next, observe that (5) not only implies that $\rho(f_0) = 0$, but that $\rho(f_0^k) = 0$ as well for any positive integer k . However, it is not immediately clear that every element of $\text{alg}(f_0)$ has spectral radius zero. However, there is an easy way to see this. Let \tilde{A} be the unitalization of A (i.e., $\tilde{A} := A \oplus \mathbb{C}$), and recall that $a \in A$ has spectral radius zero (a is called *quasi-nilpotent*) if and only if $\tilde{h}(a) = 0$ for all $\tilde{h} \in \tilde{\Delta} = \Delta(\tilde{A})$. Since each \tilde{h} is a continuous algebra homomorphism, $\ker(\tilde{h})$ is a closed ideal in \tilde{A} , and it follows that the collection of quasi-nilpotent elements is actually a *closed ideal* of A given by⁴

$$\text{rad}(A) = \bigcap_{\tilde{h} \in \tilde{\Delta}} \ker(\tilde{h}).$$

³For a reference for Tonelli's Theorem (the 'useful' version of Fubini's Theorem), see [Analysis Now, Corollary 6.6.8], or much better, see Royden's *Real Analysis*. On the other hand, if you are worried about the calculus style manipulation of limits, consider the integrand

$$F(s, t) = \begin{cases} |f(t-s)g(s)| & \text{if } 0 \leq s \leq t \leq 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

⁴This result is of interest in its own right. Note that A is always a maximal ideal in \tilde{A} , and so $\text{rad}(A)$ is always contained in A itself.

Since each f_0^k is in $\text{rad}(A)$, so is the *closed* algebra (in fact, the closed ideal) generated by f_0 . Thus, $\text{rad}(A) = A$ in this case, which is what was to be shown.

Of course, (d) is an immediate consequence of (c): if $\rho \in \Delta(A)$, then by definition there is a $f \in A$ such that $\rho(f) \neq 0$. But then $\rho(f) \geq |h(f)| > 0$, which contradicts the fact that $\text{rad}(A) = A$.

8. Here we want to give an example of a unital commutative Banach algebra A where the Gelfand transform induces an injective isometric map of A onto a proper subalgebra of $C(\Delta)$. For A , we want to take the *disk algebra*. There are a couple of ways that the disk algebra arises in the standard texts, but the most convenient for us is to proceed as follows. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. We'll naturally write \overline{D} for its closure $\{z \in \mathbb{C} : |z| \leq 1\}$, and \mathbb{T} for its boundary. Then A will be the subalgebra of $C(\overline{D})$ consisting of functions which are holomorphic on D . Using Morera's Theorem, it is not hard to see that A is closed in $C(\overline{D})$, and therefore a unital commutative Banach algebra.⁵ Notice that for each $z \in \overline{D}$, we obtain $\phi_z \in \Delta$ by $\phi_z(f) := f(z)$. We'll get the example we want by showing that $z \mapsto \phi_z$ is a homeomorphism Ψ of \overline{D} onto Δ . For convenience, let $p_n \in A$ be given by $p_n(z) = z^n$ for $n = 0, 1, 2, \dots$, and let \mathcal{P} be the subalgebra of polynomials spanned by the p_n .

- (a) First observe that Ψ is injective. (Consider p_1 .)
- (b) If $f \in A$ and $0 < r < 1$, then let $f_r(z) := f(rz)$. Show that $f_r \rightarrow f$ in A as $r \rightarrow 1$.
- (c) Conclude that \mathcal{P} is dense in A . (Hint: show that $f_r \in \overline{\mathcal{P}}$ for all $0 < r < 1$.)
- (d) Now show that Ψ is surjective. (Hint: suppose that $h \in \Delta$. Then show that $h = \phi_z$ where $z = h(p_1)$.)
- (e) Show that Ψ is a homeomorphism. (Hint: Ψ is clearly continuous and both \overline{D} and Δ are compact and Hausdorff.)
- (f) Observe that if we use the above to identify Δ and \overline{D} , then the Gelfand transform is the identity on A , and A is a proper subalgebra of $C(\overline{D})$.

⁵The maximum modulus principle implies that the map sending $f \in C(\overline{D})$ to its restriction to \mathbb{T} is an isometric isomorphism of A onto a closed subalgebra $A(D)$ in $C(\mathbb{T})$. Of course, our analysis applies equally well to $A(D)$.