LEC. 2

Math 116

choose OH's. eg: 3-5 Wed. 3-4 Mon.

Alex Barnett

1/10/06 ①

Tools for Laplace's Eqn:

$\Delta u = 0 \iff u =$ harmonic func.

Fundamental Soln.

$$\Phi(x,y) = \begin{cases} -\frac{1}{2\pi} \ln|x-y| & d=2 \\ \frac{1}{4\pi}\frac{1}{|x-y|} & d=3 \end{cases}$$

$x, y \in \mathbb{R}^d$. note: math, $x = \{x_1, x_2\}$ or $\{x_1, x_2, x_3\}$. I won't always use vector $\vec{x}$.

phys: ~~free-space~~ Green's func. as func of x, the potential due to charge at y. $x \leftrightarrow y$ symm.

← spike

Thm: $\Phi(x,y)$ harmonic in $\mathbb{R}^d \setminus \{y\}$ "the set of all points minus the single point $x=y$."

Pf: Without loss of generality, $y=0$

eg. $d=2$ $\quad \frac{\partial}{\partial x_1} \ln|x| = \frac{1}{2}\frac{\partial}{\partial x_1} \ln(x_1^2 + x_2^2) = \frac{1}{2} \cdot 2x_1 \frac{1}{x_1^2+x_2^2} = \frac{x_1}{|x|^2}$

$$\frac{\partial^2}{\partial x_1^2} \ln|x| = \frac{1}{|x|^2} + \underbrace{x_1 \frac{\partial}{\partial x_1} \frac{1}{x_1^2+x_2^2}}_{-\frac{2x_1^2}{|x|^4}}$$

$$\Delta \ln|x| = \frac{1}{|x|^2} - \frac{2x_1^2}{|x|^4} + \frac{1}{|x|^2} - \frac{2x_2^2}{|x|^4} = 0.$$

In classical notion of funcs & derivs, $\Delta_x \Phi(x,y)$ doesn't exist at $x=y$.

wrt x variable.

But can broaden our scope to include 'Schwartz distributions' (Debnath & M 6.2)

anyone?

↳ say $f(x)$ is C^∞ smooth func vanishing outside some bounded region.

Distn. is any cont. linear functional of f, ie. $g: f \to \mathbb{C}$

Dirac delta δ is a distribution: $\int f(x)\, \delta(x-a)\, dx := f(a)$

← there is no function $\delta(x)$, but we use as abbreviation.

Note: for operator $L = -\Delta$

it turns out $L_x \Phi(x,y) = \delta(x-y)$. in sense of distns, Φ is kind of inverse of L.

Ω =: domain with sufficiently smooth boundary $\partial\Omega$. u, v sufficiently smooth funcs.

Green's Thms.

(GT1) $$\int_\Omega \left(u\Delta v + \underbrace{\vec{\nabla} u}_{\text{grad } u} \cdot \vec{\nabla} v\right) dx = \int_{\partial\Omega} u \frac{\partial v}{\partial n}\, ds$$

volume; surface $= ds_x$ if $u(x) v_n(x) ds_x$

$v_n = \hat{n} \cdot \vec{\nabla} v$.

(GT2) $$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} (u v_n - v u_n)\, ds$$

contracted (lazy)

Proof: $\vec{\nabla} \cdot (u \vec{\nabla} v) = u\Delta v + \vec{\nabla} u \cdot \vec{\nabla} v$

GT1: $\int_\Omega dx$ both sides & use $\int_\Omega \vec{\nabla} \cdot (u \vec{\nabla} v)\, dx = \int_{\partial\Omega} \underbrace{\hat{n} \cdot (u \vec{\nabla} v)}_{u v_n}\, ds$

Divergence (Gauss) Thm. applied to vector field $u\vec{\nabla} v$.

GT2: subtract GT1 with $u \leftrightarrow v$ from GT1.

How smooth? Ω: To prove things (analysis of PDEs) mathematicians have litany of classes for domains eg. Ω is C — corners ok, nasty wiggles ok

C^1 — kinks ok, no corners

C^k means $x(s)$, where s parametrizes boundary, is a C^k function, ie all derivatives up to order k are continuous.

eg. Ω is 'piecewise C^∞' : smooth pieces, corners.

'Lipschitz' : locally graph of $\partial\Omega$ can be jagged but derivatives < finite const. Hölder continuous.

Div, Greens Thms work with corners, etc. (Kellogg book)

Later we will restrict to C^2 domains for integral equations to be nice.

u, v: formally Ω is an open set ($\partial\Omega$ not included). $\bar{\Omega} := \Omega \cup \partial\Omega$ 'closure' of Ω

$C^1(\Omega)$: continuous in Ω, but as approach $\partial\Omega$, may blow up $\to \infty$.

$C(\bar{\Omega})$: boundary values also continuous, and are limit of interior as $\to \partial\Omega$.

eg. $u \in C^1(\bar{\Omega})$, $v \in C^2(\bar{\Omega})$ guarantees derivs in GT1 exist classically.

Issues: Ignore for now to get going. (I am not analyst). Kress Ch 6 more careful.

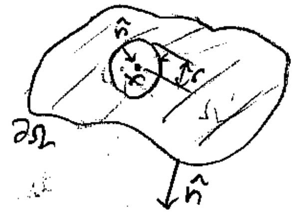
Corollary: If u harmonic, then $\int_{\partial\Omega} u_n\, ds = 0$ (zero flux), Pf. use $u=1$ in GT1

③

Green's Representation Formulas (GRF)

u harmonic in Ω

Fix $x \in \Omega$, define $\partial B(x;r)$ = circle (or sphere) radius $r > 0$ about x



as func of y, $\Phi(x,y)$ harmonic in region $R := \{y \in \Omega : |y-x| > r\}$

call this v.

Apply GT2 to region R:

$$\int_R u \underbrace{\Delta_y \Phi(x,y)}_{0} - v \underbrace{\Delta u}_{0} \, dy = \int_{\partial R} u(y) \frac{\partial \Phi}{\partial n_y}(x,y) - \frac{\partial u}{\partial n}(y)\, \Phi(x,y) \, ds_y$$

$\partial R = \partial\Omega + \partial B(x;r)$

So $$\int_{\partial\Omega} u(y) \frac{\partial \Phi}{\partial n_y}(x,y) - u_n(y)\,\Phi(x,y)\, ds_y = -\underbrace{\int_{\partial B(x;r)} u(y) \frac{\partial \Phi}{\partial n_y}(x,y)}_{a} - \underbrace{u_n(y)\,\Phi(x,y)\, ds_y}_{b}$$

Use (d=2), $$\left.\begin{aligned} \Phi(x,y) &= -\frac{1}{2\pi}\ln r \\ \frac{\partial \Phi}{\partial n_y}(x,y) &= \frac{1}{2\pi r} \end{aligned}\right\} \text{ constants, for } y \in \partial B(x;r)$$

$$a = -\frac{1}{2\pi r} \boxed{\int_{\partial B(x;r)} u(y)\, ds_y}$$

by Mean Value Thm. for integrals, and u '$\lim_{r\to 0}$' is $2\pi r\, u(x)$.

$= -u(x)$ Note, in d=3,4,... its same result, $2\pi r$ replaced by surface area d-sphere.

$$b = -\frac{1}{2\pi}\ln r \int_{\partial B(x;r)} u_n(y)\, ds_y$$ vanish by zero-flux cor. $= 0$

(GRF) $$\boxed{u(x) = \int_{\partial\Omega} u_n(y)\,\Phi(x,y) - u(y)\frac{\partial \Phi}{\partial n_y}(x,y)\, ds_y.}$$

interior values expressed as boundary integrals,

Looking ahead: integral ops $(S\sigma)(x) := \int_{\partial\Omega} \sigma(y)\,\Phi(x,y)\, ds_y$ is 'single layer potential'

surf. charge density

$(D\tau)(x) := \int_{\partial\Omega} \tau(y) \frac{\partial \Phi}{\partial n_y}(x,y)\, ds_y$ 'double - - - -'

surface dipole density

Then GRF says $u = S\sigma + D\tau$

with densities given by boundary values of u: $\tau = -u|_{\partial\Omega}$

$\sigma = +u_n|_{\partial\Omega}$

Very useful, eg. prove...

Mean Val. Thm for harmonic funcs: avg. of a harmonic func over any sphere (or ball) = value at center.
(MVT)

Proof: let Ω be open ball $\{y \in \mathbb{R}^d : |y-x| < R\}$ in GRF

$$u(x) = \int_{|y-x|=R} u(y) \underbrace{\frac{\partial \Phi}{\partial n_y}(x,y)}_{\frac{1}{2\pi r}} ds_y - \underbrace{\int u_n(y) \underbrace{\Phi(x,y)}_{= \text{const.}} ds_y}_{\text{vanishes by Zero flux Cor.}}$$

$$= \frac{1}{2\pi R} \int_{|y-x|=R} u(y)\, ds_y = \text{avg. on sphere} =: ⨍_{|y-x|=R} u \, ds$$

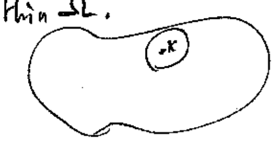
for $d \geq 3$ it's surface area

We could integrate this over $0 < r < R$ to get same average over whole ball.

Maximum Principle: max & min of harmonic func must occur on $\partial\Omega$, unless it's the const. func.

u harm. in Ω.

Proof: suppose there is a max. at some $x \in \Omega$ (ie in interior; & recall Ω open)
Then there is some sphere around x within Ω.



$\therefore$ No value on sphere can exceed $u(x)$
$\Rightarrow$ By MVT, all values must equal $u(x)$
True for all radii less than this $\Rightarrow$ u = const in that ball
Can now repeat using x = another point in that ball. $\Rightarrow$ proves u const in all of Ω.

Repeat for min values.

Cf Kress p. 61.
Note: analyticity (complex analysis) not used.

Uniqueness of interior Dirichlet BVP:
find u harmonic in Ω with $u|_{\partial\Omega} = f$ given 'boundary data'.

This has at most 1 solution:

Suppose u, v were solutions, then $u - v = 0$ on $\partial\Omega$, by Max Princ. must vanish in Ω
$\Rightarrow u = v$.

[Note: haven't proved existence – this is done via potential theory (coming up).]

Here u is a 'classical solution' ie $u \in C^2(\Omega) \cap C(\bar{\Omega})$ with $f \in C(\partial\Omega)$

There are also 'weak' solutions where derivs may not exist classically. The above is rigorous for Ω a C^2 domain (& for corners, etc).

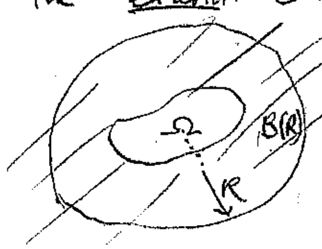
Math 116 Lecture 3

① 1/12/06

- Last time we used Maximum Principle for harmonic funcs to prove uniqueness for interior Dirichlet BVP, for classical solutions in domains for which Divergence Thm holds.

Remarks:

1) the <u>exterior</u> Dirichlet BVP in $d=3$ also can be proved unique this way:



$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \\ u|_{\partial\Omega} = f \\ u(x) = o(1) & \text{as } |x| \to \infty, \text{ uniformly in angle } \frac{x}{|x|} \end{cases}$$

'little oh', ie vanishes. The problem is not unique without this condition.

Apply the Max. Princ. to $B(R) \setminus \overline{\Omega}$, and take $R \to \infty$.

open ball of radius R

here

The difference of 2 solutions $u = u_1 - u_2$ satisfies $u|_{\partial\Omega} = 0$ and $\max_{x \in \partial B(R)} |u|$ smaller than any given constant as $R \to \infty$, $\Rightarrow u = 0$ in $B(R) \setminus \overline{\Omega}$

$\Rightarrow$ unique.

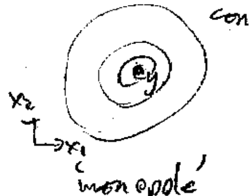
2) We have not yet proven <u>existence</u> of classical solution; one way is via integral operators (coming up!)

3) Verchota, Kenig (see Kenig. 1994 CBMS regional conference notes #83) have proven uniqueness & existence even for Lipschitz bounded domains with boundary data $f \in L^2(\partial\Omega)$. ← both $\partial\Omega$ and f can be nasty, spiky, … Very general!

This involves the idea of 'harmonic measure' & is quite advanced (I don't know it).

- Monopoles & Dipoles

$\Phi(x,y)$

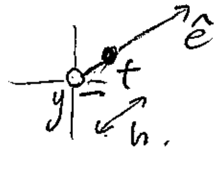


contour lines.

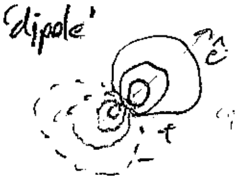
'monopole'

Place 2 such sources v. close along $\hat{e}$ unit vector:

$$\lim_{h \to 0} \frac{1}{h}\left[\Phi(x, y - h\hat{e}) - \Phi(x,y)\right]$$



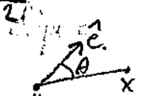
'dipole'



$$= \hat{e} \cdot \vec{\nabla}_y \Phi(x,y)$$

$$= \frac{\hat{e} \cdot (x-y)}{2\pi |x-y|^2}$$

$$= \frac{\cos\theta}{2\pi |x-y|}$$



Double layer is just setting $\hat{e} = \hat{n}_y$ with $y \in \partial\Omega$, integrating along boundary.

POTENTIAL THEORY

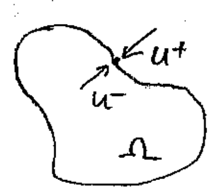
Potentials $\begin{cases} \text{Single Layer} \quad (S\sigma)(x) := \int_{\partial\Omega} \Phi(x,y)\,\sigma(y)\,ds_y & \sigma \in C(\partial\Omega) \quad \text{and} \\ \text{Double Layer} \quad (D\tau)(x) := \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_y}\,\tau(y)\,ds_y & , \ \tau \in C(\partial\Omega) \end{cases}$

are both harmonic funcs for $x \notin \partial\Omega$ (proof: integrand continuous, differentiate under integral sign).

What happens as $x \to \partial\Omega$? Sometimes depends which side you're on!

For $x \in \partial\Omega$, define $u^{\pm}(x) := \lim_{h\to 0^+} u(x \pm h\hat{n}_x)$

$$u_n^{\pm}(x) := \lim_{h\to 0^+} \hat{n}_x \cdot \vec{\nabla} u(x \pm h\hat{n}_x)$$



- Thm (Jump Relations) Let $\partial\Omega$ be class C^2, $\tau, \sigma \in C(\partial\Omega)$, $u = S\sigma$, $v = D\tau$

i) u continuous everywhere in $\mathbb{R}^d$, ie $u(x) = \int_{\partial\Omega} \Phi(x,y)\sigma(y)\,ds_y$ on $x \in \partial\Omega$

ii) $$u_n^{\pm}(x) = \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_x}\,\sigma(y)\,ds_y \ \underbrace{\mp \ \tfrac{1}{2}\sigma(y)}_{\text{jump!}}, \qquad x \in \partial\Omega$$

note x not y!

iii) $$v_n^{\pm}(x) = \int_{\partial\Omega} \frac{\partial^2\Phi(x,y)}{\partial n_x \partial n_y}\,\tau(y)\,ds_y, \qquad x \in \partial\Omega,$$ ie normal derivs. same either side $v_n^+ = v_n^-$

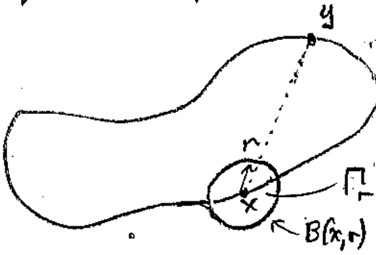
iv) $$v^{\pm}(x) = \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_y}\,\tau(y)\,ds_y \ \underbrace{\pm \ \tfrac{1}{2}\tau(y)}_{\text{jump!}}, \qquad x \in \partial\Omega$$

The above integrals are improper (since x, y both on $\partial\Omega$, integrand undefined for $x = y$) but singularities, if present, are integrable.

Eg. i) has $\begin{cases} \ln|x-y| \\ |x-y|^{-1} \end{cases}$ singularity in $\begin{cases} d=2 \\ d=3 \end{cases}$, integrable along $\begin{cases} \text{line} \\ \text{2-surface} \end{cases}$, (even if $\partial\Omega$ has corners)

Proofs are hard (Mikhlin, Colton-Kress books ~ we may get to?)

• <u>Why the jumps?</u> Heuristically...



Fix $x \in \partial\Omega$,
Split $\partial\Omega = A_r + \Gamma_r$

outside $B(x,r)$ — inside $B(x,r)$

$$u(x) = \int_{A_r} \Phi(x,y)\,\sigma(y)\,ds_y + \underbrace{\int_{\Gamma_r} \Phi(x,y)\,\sigma(y)\,ds_y}$$

well behaved as $r \to 0$, gives integral in i).

since Γ_r approx flat, in $d=2$, $\lim_{r\to 0} \frac{1}{2\pi}\int_{-r}^{r} \ln|s|\,ds = 0$
& σ bounded $\Rightarrow$ vanishes. s is arclength

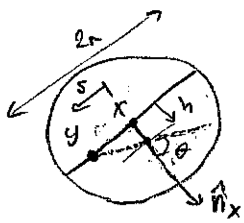
But $$u_n^+(x) = \lim_{h\to 0}\int_{A_r} \frac{\partial\Phi(x+h\hat{n}_x)}{\partial n_x}\sigma(y)\,ds_y + \lim_{h\to 0}\int_{\Gamma_r} \frac{\partial\Phi(x+h\hat{n}_x,y)}{\partial n_x}\sigma(y)\,ds_y$$

Take $r \to 0$ but $\frac{h}{r} \to 0$, ie $h \ll r$

First integral gives integral in ii)

$d=2$, s is arclength, origin at $y=x$.

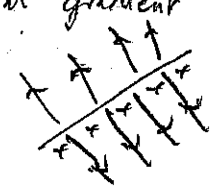
zoom in on $B(x,r)$



Approx flat, use $$\frac{\partial\Phi(x+h\hat{n}_x,y)}{\partial n_x} \approx \frac{\cos\theta}{2\pi\sqrt{h^2+s^2}} = \frac{h}{2\pi(h^2+s^2)} \leftarrow \tfrac{1}{2}\delta(s) \text{ in } \lim h\to 0.$$

$$\lim_{h\to 0} \frac{1}{2}\int_{-r}^{r} \frac{h}{\pi(h^2+s^2)}\sigma(s)\,ds = \underline{\tfrac{1}{2}\sigma(s=0)}$$

OR, For physicists, Gauss' Law gives potential gradient either side of a sheet of charge



jump in $\vec{E}$ field $(= \vec{\nabla}u)$
= charge density.

Similar argument for dipole charge sheet, gives jump in v. but same $\vec{\nabla}v$ either side

Summary of Jump relations

JR1 $\quad u = S\sigma$

JR2 $\quad u_n^{\pm} = D^T\sigma \mp \frac{1}{2}\sigma$

JR3 $\quad v_n = T\tau$

JR4 $\quad v^{\pm} = D\tau \pm \frac{1}{2}\tau$

where S, D are to be thought of as potentially singular integral operators: $C(\partial\Omega) \to C(\partial\Omega)$

D^T is D with arguments of kernel swapped.

T is deriv. of double layer op: $(T\tau)(x) = \int_{\partial\Omega} \frac{\partial^2 \Phi(x,y)}{\partial n_x \partial n_y} \tau(y)\, ds_y$, $x \in \partial\Omega$

T is more singular than D ~ I won't make any formal statement here.
(you can in Hölder spaces).

Example: Double layer with $\tau = 1$ gives constant u inside, regardless of shape of Ω!

$$\int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_y}\, ds_y = \begin{cases} -1 & x \in \Omega \\ -1/2 & x \in \partial\Omega \\ 0 & x \in \mathbb{R}^d \setminus \overline{\Omega} \end{cases}$$

Pf: Outside, harmonic in $\mathbb{R}^d \setminus \overline{\Omega}$ $\Rightarrow$ apply zero flux (why no flux contrib. at ∞? must have $u \to 0$ at ∞)
inside, use GRF in Ω with $u = -1$.
on $\partial\Omega$, use JR4 with either $u = -1$ inside or $u = 0$ outside.

You will use this to check numerical accuracy of layer potentials in HW1.

Note in $d=2$, with $\partial\Omega$ class C^2, D actually has continuous kernel.
... Surprise since $\vec{\nabla}_y \Phi(x,y)$ diverges like $O\left(\frac{1}{|x-y|}\right)$.
Prove this later.

Generally, singularity of kernel is crucial:
$d=2$, $\partial\Omega$ is 1d integral. $\int_{-1}^{1} |s|^{-\alpha} ds < \infty$ for $\alpha < 1$

A kernel $K(s,t)$ is 'weakly singular' if $|K(s,t)| \le \frac{C}{|s-t|^\alpha}$ for $\alpha < 1$

$\alpha = 0$ continuous kernel.

Let's solve interior Dirichlet BVP:

use JR4, set $v^- = f$, ask what τ is needed?

limit on inside.

Thm, if τ solves $\mathcal{D}\tau - \frac{1}{2}\tau = f$ Fredholm integral equation of 2nd kind $f \in C(\partial\Omega)$

Then $u(x) = (\mathcal{D}\tau)(x)$ is a solution to Dirichlet BVP.

proof is JR4.

MATH 116 Lecture 4

Boundary Integral Equations
Compactness
Numerical integration
MATLAB hints.

Key result from last time: construct a solution to interior Dirichlet BVP using potential theory, a double layer potential.

If τ is some function on $\partial\Omega$ solving $\underbrace{(D}_{\text{integral op.}} - \tfrac{1}{2}\underbrace{I}_{\text{identity op.}})\tau = f$ (given boundary data) $\quad (*)$

where $(D\tau)(x) := \int_{\partial\Omega} \frac{\partial \Phi(x,y)}{\partial n_y} \tau(y)\, ds_y$ is possibly improper integral if $x \in \partial\Omega$

Then $u(x) = (D\tau)(x)$, $x \in \Omega$ is a solution to $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f. \end{cases}$

This is not just an analytic tool; it will give us an efficient numerical method.

The boundary integral equation (BIE) labeled $(*)$ is a Fredholm 2^{nd} kind integral equation:

$K\tau = f$ "1^{st} kind" $\rightarrow$ nasty to invert for smoothing op K

$K\tau - \tau = f$ "2^{nd} kind" $\rightarrow$ well-behaved to invert (solve).

The 1^{st} kind is nasty since many K arising in practise are smoothing (and 'compact') in which case K^{-1} does not exist as a bounded operator (K is not 'injective')

How singular is kernel of integral op. D?

Recall $D(x,y) := \frac{\partial \Phi(x,y)}{\partial n_y} = \frac{1}{\omega_d} \frac{1}{|x-y|^{d-1}}$ for $d \geq 2$; ω_d = surf. area of d-dim unit sphere.

so for general $\partial\Omega$ with corners D is 'strongly singular', $D(x,y) = O\left(\frac{1}{|x-y|^{d-1}}\right)$

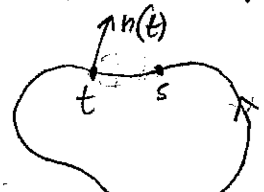
But for C^2 domain, $d \geq 2$, can bound $\hat{n}_y \cdot (x-y) \leq L|x-y|^2$ (book Colton-Kress '83)

$\Rightarrow D(x,y) \leq \frac{C}{|x-y|^{d-2}}$ which is only 'weakly singular'

Recall a 'weak' singularity is integrable (on $\partial\Omega$) but 'strong' is not: $\int_{\partial\Omega} \frac{1}{|x-y|^{\alpha}} ds_y \begin{cases} < \infty & \text{for } \alpha < d-1 \\ = \infty & \text{for } \alpha \geq d-1 \end{cases}$

for $x \in \partial\Omega$, since $\partial\Omega$ is of dimension $d-1$.

The above suggests $\mathcal{D}(x,y)$ <u>continuous</u> for C^2 domains in $d=2$. Let's prove it:



Let $t \in S^1 = [0, 2\pi)$ parametrize $\partial\Omega$ counterclockwise

$x(t) \in \mathbb{R}^2$ be boundary location

unit normal $n(t) := \dfrac{(-\dot{x}_2(t), \dot{x}_1(t))}{|\dot{x}(t)|}$ ← vector $\dot{x}$ rotated $\pi/2$ CCW.

C^2 means $\dot{x}(t)$, $\ddot{x}(t)$ cont. ($\Rightarrow$ bounded) vector funcs.

$|\dot{x}(t)| > 0 \quad \forall t$. so 'it always keeps moving'

$\Rightarrow n(t)$ also cont. vector func.

$$\mathcal{D}(s,t) = \frac{1}{2\pi}\frac{n(t)\cdot(x(s)-x(t))}{|x(s)-x(t)|^2}$$ using dipole formula for $\dfrac{\partial\Phi(x,y)}{\partial n_y}$

Note since top & bottom are continuous (& bottom nonzero) on $\{s,t \in S^1 : s \neq t\}$, so is $\mathcal{D}(s,t)$

To evaluate $\lim_{s\to t}\mathcal{D}(s,t)$ we recognize top & bottom both vanish, & so do their 1^{st} derivs.

$\Rightarrow$ l'Hôpital's rule using 2^{nd} derivs needed.

$\dfrac{\partial}{\partial s}$ top $= n(t)\cdot\dot{x}(s)$, $\dfrac{\partial^2}{\partial s^2}$ top $= n(t)\cdot\ddot{x}(s) \xrightarrow{\text{take } s=t} n(t)\cdot\ddot{x}(t)$

$\dfrac{\partial}{\partial s}$ bottom $= 2\dot{x}(s)\cdot[x(s)-x(t)]$, $\dfrac{\partial^2}{\partial s^2}$ bottom $= 2|\dot{x}(s)|^2 \xrightarrow{s=t} 2|\dot{x}(t)|^2$

$$\mathcal{D}(t,t) = \lim_{s\to t}\mathcal{D}(s,t) = \frac{1}{2\pi}\frac{n(t)\cdot\ddot{x}(t)}{2|\dot{x}(t)|^2}$$ exists, & is continuous wrt t.

$$= \frac{-1}{4\pi R(t)}$$ where $R(t)$ = local radius of curvature

$$= -\frac{\kappa(t)}{4\pi}$$ $\kappa(t)$ = curvature $= \dfrac{|\ddot{x}(t)|}{|\dot{x}(t)|^2} = \dfrac{1}{R(t)}$

($\kappa > 0$ for convex body).

So $\mathcal{D}(x,y)$ continuous ($\Rightarrow$ bounded) on $\partial\Omega\times\partial\Omega$

Thm: Let $G \in \mathbb{R}^m$ be compact set, and $K : C(G)\to C(G)$ the integral operator defined by $(K\tau)(x) = \int_G K(x,y)\tau(y)\,dy$.

If $K(x,y)$ continuous on $G\times G$ then the operator K is <u>compact</u>.

(eg. Reed & Simon v.1, any functional anal. book).

So our double layer op. $\mathcal{D}$ is compact for C^2 domains in $d=2$.

Compactness : a user's guide (v. brief — we'll do more later)

Recall a set is compact iff <u>every</u> sequence in the set contains a subsequence converging to a point <u>in that set</u>.

For subsets of $\mathbb{R}^m$ this implies closed & bounded (← note m is finite!)

Say X, Y are <u>normed</u> spaces, eg. $C([0,1])$ or $L^2(\partial\Omega)$ etc.

max norm ↑ ↑ L^2 norm

Definition:

An operator $K: X \to Y$ is compact iff for each bounded sequence $\{\varphi_n\}$ in X, the sequence $\{K\varphi_n\}$ contains a subsequence converging to an element in Y.

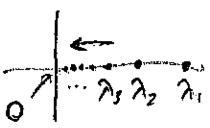
Some useful properties if K compact:

i) K is bounded operator, ie $\|K\varphi\| \le M\|\varphi\| \quad \forall \varphi \in X$

ii) Spectrum is discrete and eigenvalues tend to zero (Riesz theory).

Recall, if $K: X \to X$, $\lambda \in \mathbb{C}$ called eigenvalue if $\exists\, \varphi \in X$, $\varphi \neq 0$ st. $K\varphi = \lambda\varphi$

The 'spectrum' $\sigma(K)$ is all points where $(\lambda I - K)^{-1}$ is not bounded.



compact ops. have $\lambda = 0$ belonging to spectrum, and discrete (countably infinite) set of eigenvalues accumulating only at zero.

spectral radius = largest λ_j.

iii) Uniqueness & existence of solution φ to $K\varphi - \varphi = f$, $\forall f \in X$ holds if the homog. eqn. $K\varphi - \varphi = 0$ only has trivial solution $\varphi = 0$.

In other words K behaves 'nicely' like <u>finite</u>-dim. lin. op. (Riesz/Fredholm theory).

iv) If $\{\varphi_n\}$ is othonormal basis for $L^2(G)$, then $\|K\varphi_n\|_2 \to 0$ as $n \to \infty$.
(We've specialized to $K: L^2(G) \to L^2(G)$).

Surprising result! K is smoothing.
Eg. $\varphi_n = \sin nx$ on $L^2[0,2\pi]$.

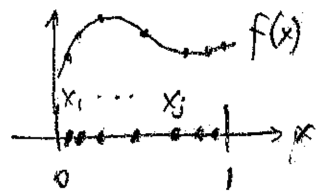
Note that ii) means K^{-1} is unbounded. ⇝ bad idea to invert K numerically.

We also have : Thm : integral operators with weakly singular kernels are compact

Note that iii) will allow <u>existence</u> of solution to Dirichlet interior BVP to be proved.

∼ Beautiful proofs ∼ more later ...

Numerical approximation of integrals



$$\int_0^1 f(x)\,dx \approx \sum_{j=1}^{M} w_j\, f(x_j)$$

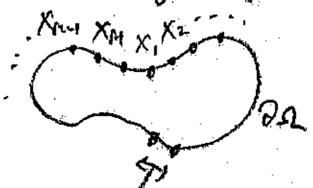
weights — nodes or quadrature points.

there are many schemes, aiming for
i) high accuracy, ie small error, for certain classes of f,
ii) high-order convergence, ie error $= O(M^{-p})$ p = order large.

Eg see Numerical Recipes. Note i) & ii) not always compatible!

For now you care about $\int_{\partial\Omega} f(x)\,dx$ in $d=2$, ie smooth function on periodic domain (closed curve)

We will get good results using equal weights w_j and equally-spaced (in arclength) x_j.



w_j = arclength Δs between nodes, $\forall j$.

- Order of convergence then depends on smoothness of f
- It can be shown if f is analytic function, then convergence is exponential, ie exceeds any order p!

error $= O(e^{-\alpha M})$ ← α related to distance f can be continued analytically into a strip around real axis.

In HW1 you'll find it convenient to parametrize by angle θ not arclength s

Then $\int_{\partial\Omega} f(x)\,ds_x = \int_0^{|\partial\Omega|} f(s)\,ds = \int_0^{2\pi} \underbrace{f(\theta)\frac{ds}{d\theta}}\,d\theta$

arclength; change variable.

therefore you may choose $w_j = \left.\frac{ds}{d\theta}\right|_{x_j} \cdot \Delta\theta$ ← new weights.

thus if $\frac{ds}{d\theta}$ is as smooth as f, you retain same convergence.

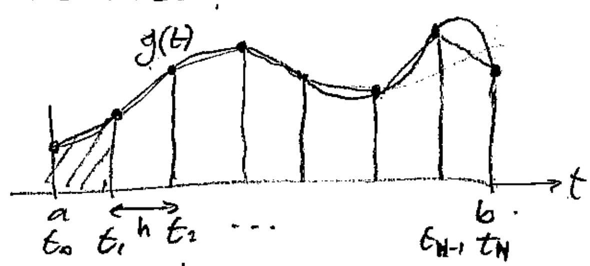
Next time we'll apply this to solve BIE: Nyström method.

Math 116 Lecture 5

1/19/06
Barnett

Numerical integration: more about quadrature

TRAPEZOID RULE



Given $g(t)$ function,
Want integral over closed interval $[a,b]$
equally-spaced points labeled $j = 0 \dots N$
spacing $h = \frac{b-a}{N}$, $t_j = a + hj$

$$\int_a^b g(t)\,dt \approx h\left[\tfrac{1}{2}g(t_0) + g(t_1) + g(t_2) + \dots + g(t_{N-1}) + \tfrac{1}{2}g(t_N)\right]$$

this is just sum of areas of trapezoids

What is order of convergence?

Define error (remainder) $R[g] := \int_a^b g(t)\,dt - h\left[\tfrac{1}{2}g(t_0) + g(t_1) + \dots + g(t_{N-1}) + \frac{g(t_N)}{2}\right]$

Intuitively, if g is 'smooth' then area error for each trapezoid is

like segment of circle, radius $R \sim (g'')^{-1}$

Estimate area $h\,\Big[$ $\theta \approx \frac{h}{R}$ $\approx$ similar tri's $\frac{h^2}{R}$ area $\sim \frac{h^3}{R} \sim h^3 g''$.

$\Rightarrow$ total error $\sim N h^3 g''$
$= O\left(\frac{1}{N^2}\right) g'' = O(h^2)\, g''$.

This argument shows it's 2nd order.

Thm: Let $g \in C^2[a,b]$, then $|R[g]| \le \frac{1}{12} h^2 (b-a) \|g''\|_\infty$

Proof: consider region $[t_0, t_1]$, define 'Peano' kernel here $k(t) := \frac{(t-t_0)(t_1-t)}{2}$

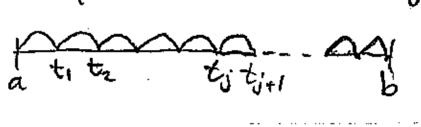
Then
$$\int_{t_0}^{t_1} k(t)\, g''(t)\,dt \underset{\substack{\text{by parts,}\\ k(t_0)=k(t_1)=0.}}{=} -\int_{t_0}^{t_1} k' g'\,dt \underset{\text{by parts}}{=} -\Big[k' g\Big]_{t_0}^{t_1} + \int_{t_0}^{t_1} k'' g\,dt$$

$k' = t_1 + t_0 - 2t$ $\quad\quad$ $k'' = -1$

$$= h\,\frac{g(t_0) + g(t_1)}{2} - \int_{t_0}^{t_1} g(t)\,dt$$

Summing over all intervals $[t_j, t_{j+1}]$ gives, with $k(t) := \frac{1}{2}(t-t_j)(t_{j+1}-t)$ for $t_j \le t \le t_{j+1}$

$$\int_a^b k(t)\, g''(t)\,dt = -R[g]$$



since $k(t)$ nonnegative on $[a,b]$

$$|R[g]| \le \left|\int_a^b k\, g''\, dt\right| \le \underbrace{\|k\|_1}\, \|g''\|_\infty$$

$$\int_a^b k(t)\,dt = N\int_0^h \tfrac{1}{2}t(h-t)\,dt = \frac{N}{2}\left(\frac{h^3}{2}-\frac{h^3}{3}\right) = h^2(b-a)/12$$

Remarks: • this quadrature rule is in the form $\sum_{j=1}^{N} w_j\, g(t_j)$

- in some sense the order $O(N^{-2})$ is due to treatment of the ends of interval. (it is possible to get higher-order with more complicated weights near ends). eg. Simpson, Gaussian quad, ... beautiful.
- We really care about periodic intervals, where there are no end effects. As mentioned, for smooth (analytic) functions, a simple equal-spaced, equal-weight scheme gives exponential $O(e^{-\kappa M})$ convergence! Let's postpone proofs to another lecture.

NYSTRÖM METHOD $\rightsquigarrow$ apply quadrature rule to solve integral equation

$$(K - I)\tau = f \qquad \text{Fredholm } 2^{nd} \text{ kind.}$$

$$\underbrace{\int K(s,t)\,\tau(t)\,dt}_{\text{quad.}\ \approx\ \sum_{j=1}^{N} w_j K(s,t_j)\,\tau(t_j)} - \tau(s) = f(s) \qquad \text{holds for all } s \in \text{domain (left general).}$$

Must hold at each $s = t_i$:

$$\sum_{j=1}^{N} \underbrace{w_j\, K(t_i,t_j)}_{\text{call } (\tilde{K})_{ij}}\ \underbrace{\tau(t_j)}\ - \tau(t_i) = \underbrace{f(t_i)} \qquad \text{for } i = 1\cdots N.$$

$N\times N$ matrix — j'th component of solution vector $\vec{\tau} \in \mathbb{R}^N$ — ith comp. of vector $\vec{f} \in \mathbb{R}^N$

$$\Rightarrow \quad [\tilde{K} - I]\,\vec{\tau} = \vec{f}$$

linear algebra problem... takes $O(N^3)$ CPU effort.

Nyström's key observation was that the best way to find $\tau(t)$ inbetween the t_j was:

$$\tau(t) = \sum_{j=1}^{N} w_j K(t,t_j)\,\tau(t_j) - f(t)$$

, ie to use the kernel itself to interpolate.

If we had done this method to a 1^{st}-kind I.E., would have got,

$\tilde{K}\vec{\tau} = \vec{F}$ (note the final interpolation step not possible here)

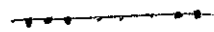
Eg. integral kernel $K(s,t) = e^{-(5|s-t|)^2}$ on $[0,1]$

$\in C[0,1]^2$

width $\sim \frac{1}{5}$.

use $w_j = \frac{1}{N} \quad \forall j$

$t_j = \frac{\frac{1}{2}+j}{N}$

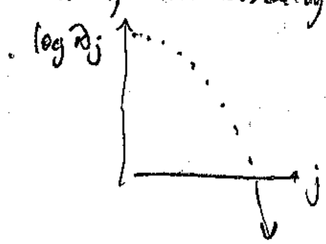


Gives for $\tilde{K}$ matrix exactly A matrix, $a_{ij} = \frac{1}{N} e^{-\left(\frac{5(i-j)}{N}\right)^2}$

from HW1.1

Since K continuous, op $K: C[0,1] \to C[0,1]$ is compact.

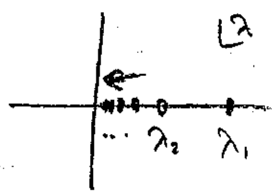
How manifest itself numerically?



eigenvalues continue down towards zero (exponentially fast) as $N \to \infty$.

reflects itself in ill-conditioned solution of $A\vec{x} = \vec{b}$

Spectrum in $\mathbb{C}$:



Now you see why 2^{nd}-kind are better. ; $(A - I)\vec{x} = \vec{b}$ is well-conditioned.

Math 116. Lecture 6

ERROR ANALYSIS of INTEGRATION OF PERIODIC FUNCS.

Why is crude equal-weight equally-spaced quadrature $\int_0^{2\pi} g(x)\,dx \approx \underbrace{\frac{2\pi}{N}}_{w_j \text{ all equal.}}\sum_{j=1}^{N} g\left(\frac{2\pi j}{N}\right)$ so good?

ANALYTIC CASE (§9.4, Kress, "Numerical Analysis").

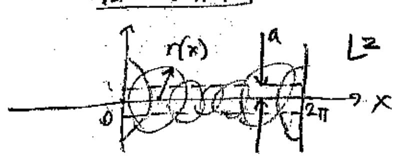
Thm. Let $g : \mathbb{R} \to \mathbb{R}$ be analytic & 2π-periodic. Then there exists a strip $D = \mathbb{R} \times (-a,a) \subset \mathbb{C}$ with $a>0$ s.t. g can be extended to a holomorphic and 2π-periodic bounded function $g : D \to \mathbb{C}$.
The error for above quadrature rule is bounded by

$$|R_N[g]| \leq \frac{4\pi M}{e^{Na}-1}$$

where M is a bound for holomorphic function g on D.

Remark: this proves exponential convergence of errors $O(e^{-aN})$

Proof: $\boxed{1^{st} \text{ PART}}$



Analytic ⇒ at each $x \in \mathbb{R}$, Taylor expansion converges in some open disk radius $r(x) > 0$.
This provides a 2π-periodic holomorphic extension of g.
since x & $x+2\pi$ have same Taylor expansion.

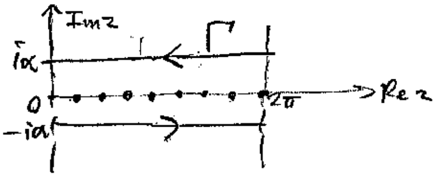
Can cover $[0,2\pi]$ with finite # of such disks.
a can be chosen to be any width $<$ minimum $r(x)$.
g is then bounded on the strip D.

$\boxed{2^{nd} \text{ PART}}$

Consider $\cot(z)$, which has residuals (pole strengths) of 1 at $z_j = \pi j$, $j \in \mathbb{Z}$ (since $\frac{d}{dz}\Big|_0 \tan z = 1$)

Then $g(z)\cot\left(\frac{N}{2}z\right)$ has residuals $\frac{2}{N}\, g\left(\frac{2\pi j}{N}\right)$ at points $z_j = \frac{2\pi j}{N}$

②



Residue thm gives, for $\alpha < a$,

$$\int_\Gamma g(z)\cot\left(\frac{Nz}{2}\right)dz = 2\pi i \sum \text{residuals}$$

$$= \frac{4\pi i}{N}\sum_{j=1}^{N} g\left(\frac{2\pi j}{N}\right) \qquad (*)$$

note periodic $\Rightarrow$ doesn't need to close.

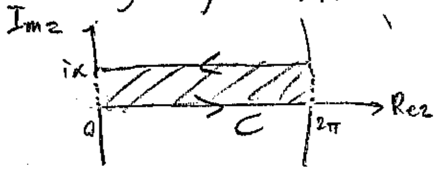
Schwarz reflection principle: $\begin{cases} g \text{ real on } \mathbb{R} \text{ so } g(\bar{z}) = \overline{g(z)} \\ \text{ie imaginary part is antisymmetric in Im } z. \end{cases}$

$\Rightarrow$ LHS integral becomes $-i\int_{i\alpha}^{i\alpha+2\pi} 2\,\mathrm{Im}\, g(z)\cot\left(\frac{Nz}{2}\right)dz = 2i\,\mathrm{Re}\int_{i\alpha}^{i\alpha+2\pi} i\, g(z)\cot\left(\frac{Nz}{2}\right)dz$

Using $(*)$, $\mathrm{Re}\int_{i\alpha}^{i\alpha+2\pi} i\cot\left(\frac{Nz}{2}\right) g(z)\,dz = \frac{2\pi}{N}\sum_{j=1}^{N} g\left(\frac{2\pi j}{N}\right)$

our quadrature rule!

Cauchy integral thm:



$\oint_C g(z)\,dz = 0$ since analytic in D

true integral

so $\mathrm{Re}\int_{i\alpha}^{i\alpha+2\pi} g(z)\,dz = \int_0^{2\pi} g(x)\,dx$

$\Rightarrow$ error $R_N[g] = \mathrm{Re}\int_{i\alpha}^{i\alpha+2\pi} \underbrace{\left[1 - i\cot\left(\frac{Nz}{2}\right)\right]}_{\text{bounded?}} g(z)\,dz$

$|g(z)|$ on $x+i\alpha$ is bounded by M.

$$\left|1 - i\cot\frac{N(x+i\alpha)}{2}\right| = \left|1 + \frac{e^{i\frac{Nx}{2}}e^{-\frac{N\alpha}{2}} + e^{i\frac{Nx}{2}}e^{\frac{N\alpha}{2}}}{e^{i\frac{Nx}{2}}e^{-\frac{N\alpha}{2}} - e^{i\frac{Nx}{2}}e^{\frac{N\alpha}{2}}}\right| \le \frac{2}{e^{N\alpha}-1}$$

Take limit $\alpha \to a$. QED.

Remark: $\frac{1}{\pi N}\,\mathrm{Im}\cot\frac{Nz}{2}$ is just an approximation to double layer potential placed along the Re axis $\approx +1$, ≈ -1. cool!

There also exist Euler-Maclaurin theorems for C^{2m+1} FUNCTIONS:

Thm: Let $g \in C^{2m+1}$ be 2π-periodic, for some $m \ge 1$.
Then $|R_N[g]| \le \frac{C}{N^{2m+1}}\int_0^{2\pi}|g^{(2m+1)}(x)|\,dx$ where $C = 2\sum_{k=1}^{\infty}\frac{1}{k^{2m+1}}$

Proof requires Bernoulli poly's (see Kress §9.4). $\boxed{\text{Smoother } g \Rightarrow \text{higher-order convergence}}$

We can understand these convergence orders using result that Fourier series of $\begin{cases}\text{analytic} \\ C^m\end{cases}$ func die $\begin{cases}\text{exponentially in } k \\ \text{like } \frac{1}{k^{m+1}}\end{cases}$ ← Fourier variable $k = 0, 1, 2 \dots$

and the quadrature rule approximates the zeroth Fourier coefficient with errors involving Fourier coefficients $\pm N$, $\pm 2N$, etc. (cf. Nyquist sampling thm).

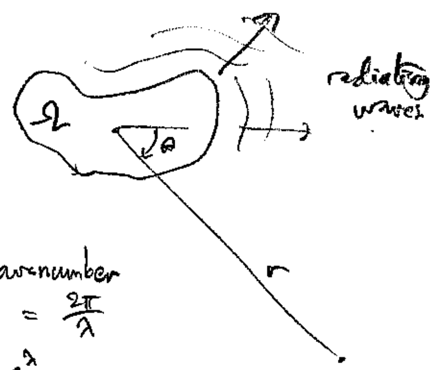
Scattering theory

Exterior Helmholtz problem

$$(\Delta + k^2) u^s = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega} \qquad d=$$

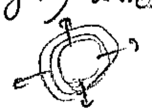
$$u^s = f \quad \text{on } \partial\Omega$$

$$\frac{\partial u^s}{\partial r} - iku^s = o\left(r^{\frac{1-d}{2}}\right) \quad \text{Sommerfeld radiation condition}$$



wavenumber $k = \frac{2\pi}{\lambda}$

says: only outward-going waves persist at large distances.



We will show, given $f|_{\partial\Omega}$, the above has unique solution u^s

Scattering: if u^i is incident field. (eg. $u^i(x) = e^{ik\hat{d}\cdot x}$) plane wave; free-space solution $(\Delta + k^2) u^i = 0$ in $\mathbb{R}^d$

choose $f = -u^i$ on $\partial\Omega$

then total field $u = u^i + u^s$

obeys $\begin{cases}(\Delta + k^2) u = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega} \\ u = 0 & \text{on } \partial\Omega\end{cases}$ ← Dirichlet reflecting BCs.

Fundamental solutions

$$\Phi(x,y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|) & d=2 \\ \frac{e^{ik|x-y|}}{4\pi|x-y|} & d=3 \end{cases}$$

$H_0^{(1)}$ is Hankel func $= J_0 + iY_0$

Math 116 LECTURE 7

Scattering Theory

<u>Wave equation</u> $U(x,t)$ ($x \in \mathbb{R}^d$, time-dependent.) e.g. acoustic pressure field. (real-valued)

$$U_{tt} = c^2 \Delta U \quad \text{in } \mathbb{R}^d \qquad \text{(WE)}$$

c = wave speed

see Whitham book, physics, etc.

Time-harmonic $U(x,t) = \mathrm{Re}\left[e^{-i\omega t} u(x)\right]$ ω = (angular) frequency (1)

rotating exponential: $e^{-i\omega t}$; $u(x)$: complex, stationary.

Subst. (1) in (WE): $\mathrm{Re}\,(-i\omega)^2 e^{-i\omega t} u = c^2 \mathrm{Re}\, e^{-i\omega t} \Delta u \quad \forall t$

$$\Rightarrow \quad (\Delta + k^2) u = 0 \qquad \text{Helmholtz Eqn.}$$

wavenumber $k = \frac{\omega}{c}$.

$\boxed{\text{Flux}}$: flow of energy

(WE) obeys a beautiful <u>conservation law</u>: any bounded domain $\Omega \subset \mathbb{R}^d$ $\rightsquigarrow$

$$\frac{\partial}{\partial t} \int_\Omega E(x,t)\, dx = -\int_{\partial\Omega} \hat{n}_y \cdot \vec{F}(y,t)\, ds_y$$

E: energy density (or prob. density if you're a quantum mechanic); $\vec{F}$: flux vector

"rate of change of energy = flux leaving region."

How?

Mult. WE by U_t:

$$\underbrace{U_t U_{tt}}_{\frac{1}{2}(U_t^2)_t} = c^2 \underbrace{U_t \Delta U}_{\vec{\nabla}\cdot(U_t \vec{\nabla} U) \,-\, \underbrace{\vec{\nabla} U_t \cdot \vec{\nabla} U}_{\frac{1}{2}|\vec{\nabla} U|^2_t}}$$

by calculus

$$\text{so} \quad \underbrace{\frac{\partial}{\partial t} \frac{1}{2}\big(\underset{\text{kinetic}}{U_t^2} + \underset{\text{potential (elastic)}}{c^2 |\vec{\nabla} U|^2} \big)}_{\text{defines } E(x,t)} = -\vec{\nabla}\cdot \underbrace{\left(-c^2 U_t \vec{\nabla} U\right)}_{\text{defines } \vec{F}(x,t)}$$

$\rightsquigarrow$ cons. Law in "differential form" $E_t + \mathrm{div} \vec{F} = 0$

Integrate over any Ω and apply Divergence Thm proves the conservation law (integral form).

What is flux for static field $u(x)$?

Energy can oscillate in & out of region $\Rightarrow$ $\vec{f}(x) := \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \vec{F}(x,t)\, dt =: \langle \vec{F}(x,t) \rangle$

$\vec{f}$: net flux

integral over one period (or time-average).

$$\vec{F}(x,t) := -c^2\, U_t \vec{\nabla} U = -c^2\, \mathrm{Re}\left[-i\omega e^{-i\omega t} u\right] \mathrm{Re}\left[e^{-i\omega t}\vec{\nabla} u\right]$$

use $\mathrm{Re}\, a = \frac{a+\bar{a}}{2}$

$$= -\frac{\omega c^2}{4}\Big[\underbrace{-i\cancel{e^{-2i\omega t}} u\vec{\nabla} u}_{\text{vanish on time average}} + \underbrace{i\bar{u}\vec{\nabla} u - iu\vec{\nabla}\bar{u}}_{2\,\mathrm{Im}\, u\Delta\bar{u}} + \underbrace{i\cancel{e^{+2i\omega t}}\bar{u}\vec{\nabla}\bar{u}}_{\text{vanish on time average}}\Big]$$

so $\vec{f}(x) = \underbrace{-\frac{\omega c^2}{2}}_{\text{we could drop the constant. (it's } \frac{\hbar}{m} \text{ in quantum mech.)}} \cdot\ \mathrm{Im}\left[u\vec{\nabla}\bar{u}\right]$

Note $\langle E(x,t)\rangle = \frac{1}{2}|u|^2 + \frac{E^2}{2}\underbrace{|\vec{\nabla} u|^2}_{\text{means } \vec{\nabla} u\cdot\vec{\nabla}\bar{u}}$ ← this is (square of) H^1 Sobolev "energy" norm.

$\boxed{\text{Radiation condition}}$:

Solution u^s to Helmholtz Eqn. in some region including exterior of some large sphere is 'radiating' if

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

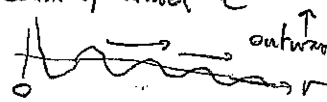
holds uniformly in all directions $\frac{x}{r}$, $r := |x|$.

Sommerfeld condition (1912)

Ensures all flux is outward

Eg. $d=3$, $u^s = \frac{e^{ikr}}{r}$ is solution in $\mathbb{R}^3\setminus\{0\}$

Direction of travel $e^{i(kr-\omega t)}$ ↑ outward.



$$r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = r\left(\frac{-e^{ikr}}{r^2} + \cancel{ik\frac{e^{ikr}}{r}} - \cancel{ik\frac{e^{ikr}}{r}}\right) \to 0 \quad \text{as } r\to\infty.$$

So $\frac{e^{ikr}}{r}$ is radiating but $\frac{e^{-ikr}}{r}$ is not!

This condition ensures (proof: Sommerfeld, see Kress) uniqueness for

Exterior Helmholtz BVP
$$\begin{cases} (\Delta + k^2)u^s = 0 & \text{in } \mathbb{R}^d\setminus\bar{\Omega} \\ u^s = f & \text{on } \partial\Omega \\ \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 & \text{uniformly in angle.}\end{cases}$$

"what is field due to radiating body"?

$\boxed{\text{Scattering problem}}$:

Given incoming wave u^i, what $u = u^i + u^s$ solves

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in } \mathbb{R}^d\setminus\bar{\Omega} \\ u = 0 & \text{on } \partial\Omega \\ u^s \text{ is radiating.} \end{cases}$$

Dirichlet, or 'sound-soft' BCs.

u^i solves Helmholtz itself
⇒ Scatt. prob. solved by finding u^s solution to ext. Helmholtz with $f = -u^i$ on $\partial\Omega$

$\boxed{\text{Fundamental Solution}}$

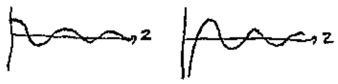
recall $-(\Delta_x + k^2)\,\Phi(x,y) = \delta(x-y)$, as distributions.

$$\Phi(x,y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|) & d=2 \\ \dfrac{e^{ik|x-y|}}{4\pi|x-y|} & d=3 \end{cases}$$

radiating solution to Helmholtz in $\mathbb{R}^d \setminus \{y\}$

Remarks

i). $\underset{\text{Hankel}}{H_0^{(1)}(z)} := \underset{\text{Bessel}}{J_0(z)} + i\underset{\text{Neumann}}{Y_0(z)} \sim \sqrt{\frac{2}{\pi z}}\, e^{i(z-\pi/4)} \left\{ 1 + O\left(\tfrac{1}{z}\right) \right\}$ as $z \to \infty$



wave, decaying ampl. like $r^{-1/2}$

For $r \to 0$ in $d=2$ note $\Phi(x,y) = \frac{-1}{2\pi} \ln|x-y| + \underbrace{O(1)}_{\text{bounded}}$

same singularity as fund. soln. for Laplace's eqn. (clearly true in $d=3$ too).

Since singularity same, can show all Jump Relations are same as before.

ii) Your computer knows $H_0^{(1)}(z)$ (math libraries, Matlab, etc.).

$\boxed{\text{Greens Repr. Formula:}}$ Let u be a Helmholtz solution, then

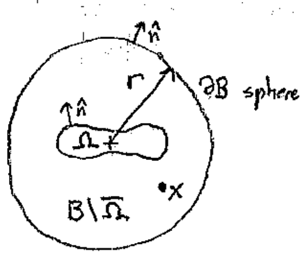
$$u(x) = \pm \int_{\partial\Omega} \left[u_n(y)\,\Phi(x,y) - u(y) \frac{\partial \Phi(x,y)}{\partial n_y} \right] ds_y \qquad \text{for } x \in \begin{cases}\text{inside } (+) \\ \text{outside } (-)\end{cases}$$

- Interior GRF: $x \in \Omega$, u_n means u_n^- ie from inside.

<u>Proof</u>: same as GRF for Laplace operator, using fact that singularity is the same.

- Exterior GRF: $x \in \mathbb{R}^d \setminus \overline{\Omega}$, u_n means u_n^+ from outside, u must be <u>radiating</u> solution. The radiation condition ensures there's no 'boundary term' at ∞.

<u>Proof</u>:



First show $\int_{\partial B} |u|^2\, ds = O(1)$ $\quad r \to \infty$

$$\underbrace{\int_{\partial B} \left| \frac{\partial u}{\partial r} - iku \right|^2 ds}_{\text{vanishes if radiating, as } r\to\infty} \overset{\text{expand}}{=} \int_{\partial B} \left|\frac{\partial u}{\partial r}\right|^2 + k^2|u|^2 + 2k\,\underbrace{\mathrm{Im}\, u \frac{\partial \bar{u}}{\partial r}}_{\text{flux leaving sphere}}\, ds \qquad (2)$$

But, in any region R in which u a solution, have <u>flux balance</u> (no net flux), since

$$\mathrm{Im} \int_{\partial R} u\, \bar{u}_n\, ds = \mathrm{Im} \int_R \underbrace{u \underbrace{\Delta \bar{u}}_{-k^2\bar{u}} + \nabla u \cdot \nabla \bar{u}}_{\text{purely real}}\, dx \qquad \text{by GT1}$$

$$= 0$$

(t)

Apply flux balance to $R = B\setminus\overline{\Omega}$ gives $\int_{\partial B} 2k\,\mathrm{Im}\, u\frac{\partial \bar{u}}{\partial r}\, ds = \underbrace{\int_{\partial\Omega} 2k\,\mathrm{Im}\, u\bar{u}_n\, ds}_{\text{some finite number, } F}$

Combine with (2) gives $\lim_{r\to\infty}\int_{\partial B} \left|\frac{\partial u}{\partial r}\right|^2 + k^2|u|^2\, ds = -F$

← sum of nonnegative terms so each must be bounded

$\Rightarrow \lim_{r\to\infty}\int_{\partial B}|u|^2 ds = O(1)$.

Now Take sphere surface term in GRF, show vanishes as $r\to\infty$:

$$\int_{\partial B}\left[u(y)\frac{\partial\Phi(x,y)}{\partial n_y} - u_n(y)\,\Phi(x,y)\right] ds_y = \underbrace{\int_{\partial B} u\left[\frac{\partial\Phi}{\partial n_y} - ik\Phi\right] ds_y}_{=:I_1} - \underbrace{\int_{\partial B}\Phi\left[u_n - iku\right] ds_y}_{=:I_2}$$

↖ for $x\in B\setminus\overline{\Omega}$

I_1 & I_2 vanish as $r\to\infty$:

"little oh".

$$\frac{\partial\Phi(x,y)}{\partial n_y} - ik\Phi(x,y) = o\left(\frac{1}{r^{\frac{d-1}{2}}}\right) \quad \text{since } \Phi(x,\cdot) \text{ radiating}$$

Schwarz

$$I_1 \le \underbrace{\sqrt{\int_{\partial B}|u|^2 ds}}_{O(1)}\;\underbrace{\sqrt{\int\left[\frac{\partial\Phi}{\partial n_y} - ik\Phi\right]^2 ds}}_{o(1)} \longrightarrow 0 \quad \text{as } r\to\infty$$

since surface area is $C r^{d-1}$

$\Phi(x,\cdot) = O\left(\frac{1}{r^{\frac{d-1}{2}}}\right)$ and u radiating $\Rightarrow$ $I_2\to 0$ as $r\to\infty$ (integrand uniformly $\to 0$)

minus since normal direc.

Finally, applying Interior GRF to $B\setminus\overline{\Omega}$ gives $u(x) = -\int_{\partial\Omega+\partial B} u_n(y)\Phi(x,y) - u(y)\frac{\partial\Phi(x,y)}{\partial n_y}\, ds_y$

Take $\lim r\to\infty$, QED. ↗ bounded domain $\quad x\in B\setminus\overline{\Omega}$ $\quad$ ↖ just shown vanishes as $r\to\infty$

This was proved by Wilcox (1956) ... see Colton & Kress "Inverse..." book Thm. 2.4.

Boundary Integral Eqns:

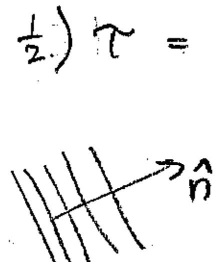
The crude way to solve exterior Helmholtz BVP is pure double-layer representation:

$x\in\mathbb{R}^d\setminus\overline{\Omega}$, $\quad u^s(x) = (D\tau)(x)$ $\quad$ | τ some density on $\partial\Omega$

JR4 $\quad u^+ = D\tau + \frac{1}{2}\tau$ $\quad$ we want $u^+ = f = -u^i|_{\partial\Omega}$ incident field.

Therefore u solves scattering problem if $\left(D+\frac{1}{2}\right)\tau = -u^i|_{\partial\Omega}$

Typically $u^i(x) = e^{ik\hat{n}\cdot x}$, a plane wave

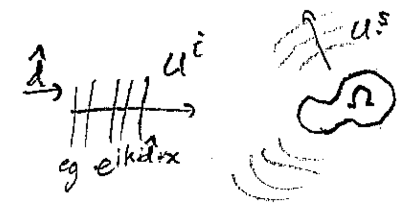


Next time: why does $D+\frac{1}{2}$ go singular at some k?

MATH 116 - LECTURE 8.

①. 1/31/06.

FAR FIELD:



total field $u = u^i + u^s$

k = wavenumber.

Recall scattering solved by finding u^s solving Exterior Dirichlet BVP for Helmholtz eqn:

$$\begin{cases} (\Delta + k^2) u^s = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega} \\ u^s = -u^i & \text{on } \partial\Omega \\ \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \end{cases}$$

although didn't prove it, has unique soln for C^2 domains, $u^i \in C(\partial\Omega)$

radiation condition

One way to measure u^s is by its 'far-field pattern' $u_\infty(\hat{x})$: direction $\in S^{d-1}$

Thm: every radiating soln. to Helmholtz eqn. has asymptotic behavior of outgoing spherical wave

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{\frac{d-1}{2}}}\left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\} \quad \text{as } |x| \to \infty.$$

Proof. ($d=3$ case)

Fund sol. $\Phi(x,y) = \dfrac{e^{ik|x-y|}}{4\pi|x-y|}$ in $d=3$

Note $|x-y| = \sqrt{|x|^2 - 2x\cdot y + |y|^2} = |x| - \hat{x}\cdot y + O\left(\frac{1}{|x|}\right)$

so $$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|}\left\{ e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right)\right\}$$

$$\frac{\partial}{\partial n_y}\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|}\left\{ \frac{\partial}{\partial n_y} e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right)\right\}$$

Insert these into GRF, proved last time for radiating solutions:

$$u(x) = \frac{e^{ik|x|}}{|x|}\left\{ \underbrace{\frac{1}{4\pi}\int_{\partial\Omega}\left[u(y)\frac{\partial}{\partial n_y}e^{-ik\hat{x}\cdot y} - u_n(y)e^{-ik\hat{x}\cdot y}\right] ds_y}_{\text{identify as } u_\infty(\hat{x}) \text{ in Thm.}} + O\left(\frac{1}{|x|}\right)\right\}$$

For $d=2$ we use $\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|)$ with $H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{i(z-\frac{\pi}{4})}\left(1 + O\left(\frac{1}{z}\right)\right)$ as $z\to\infty$

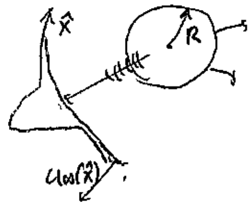
Similar proof to above gives,

$$u(x) = \frac{e^{ik|x|}}{\sqrt{|x|}}\cdot\left\{ \overbrace{\frac{e^{i\pi/4}}{\sqrt{8\pi k}}\int_{\partial\Omega}\left[u(y)\frac{\partial}{\partial n_y}e^{-ik\hat{x}\cdot y} - u_n(y)e^{-ik\hat{x}\cdot y}\right] ds_y}^{\text{identify as } u_\infty(\hat{x})} + O\left(\frac{1}{|x|}\right)\right\}$$

Interpretation : $u_\infty(\hat{x})$ obtained by (weighted) integrals of u & u_n on $\partial\Omega$.

Outgoing flux to ∞ is $\sim \int_{\partial B} \mathrm{Im}[\bar{u}\,\vec{\nabla}\vec{u}]\,ds \sim R^{d-1} \int_{S^{d-1}} \mathrm{Im}\left[\bar{u}^s \frac{\partial u^s}{\partial R}\right] d\hat{x}$

use far field rep. $\frac{1}{R^{\frac{d-1}{2}}} \bar{u}_\infty \cdot \frac{k}{R^{\frac{d-1}{2}}} u_\infty$



$$\sim \int_{S^{d-1}} |u_\infty(\hat{x})|^2 \, d\hat{x}$$

integral of power radiated over all angles.

Given double-layer rep. for u^s, how do you find u_∞?

$$u^s(x) = (D\tau)(x) \qquad \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}$$

density func.

recall τ found by BIE, $(D+\frac{1}{2})\tau = -u^i$ on $\partial\Omega$.

As above, consider $|x| \to \infty$:

$$u^s(x) = \int_{\partial\Omega} \frac{\partial \Phi(x,y)}{\partial n_y} \tau(y)\, ds_y = \frac{e^{ik|x|}}{\sqrt{|x|}} \Bigg\{ \underbrace{\frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial\Omega} \tau(y) \overbrace{\frac{\partial}{\partial n_y} e^{-ik\hat{x}\cdot y}}^{+ik\,\hat{n}_y\cdot\hat{x}\, e^{-ik\hat{x}\cdot y}} ds_y}_{u_\infty} + O\left(\frac{1}{|x|}\right) \Bigg\}$$

$$\boxed{\text{ie } u_\infty(\hat{x}) = \frac{e^{-i\pi/4}}{\sqrt{8\pi k}} \cdot ik \int \tau(y)\, (\hat{n}_y \cdot \hat{x})\, e^{-ik\hat{x}\cdot y}\, ds_y.}$$

this is (3.6) in Kress 1991 review, for $\eta = 0$ case

In practise, once you have τ at the boundary points, mult. each by the geometric factor $(\hat{n}_y \cdot \hat{x})\, e^{-ik\hat{x}\cdot y}$ and use same quadrature as usual.

Interior resonance problem:

Consider interior eigenvalue problem $\begin{cases} -\Delta u = k^2 u & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$

non-trivial u_j = eigenfunction
k_j = 'eigenwavenumbers' (or just eigenvalues). $j = 1, 2, \cdots \infty$

u_j solve Helmholtz eqn. $(\Delta + k_j^2) u_j = 0$ in Ω

So they could be represented by single-layer potential $u(x) = (S\sigma)(x)$, $x \in \Omega$.

JR2 then says, $u_n^- = D^T\sigma + \frac{1}{2}\sigma$ for limiting value just inside boundary.
adjoint of D (kernel has x, y swapped).

Neumann BCs mean LHS is zero $\Rightarrow$ $(D^T + \frac{1}{2})\sigma = 0$ for some nonzero σ.

The operator $D^T + \frac{1}{2}$ has <u>nontrivial nullspace when $k = k_j$</u>
in $C(\partial\Omega) \to C(\partial\Omega)$.
actually a good method to find eigenmodes.

Fredholm theory gives us. $\dim \mathrm{Nul}(I - D) = \dim \mathrm{Nul}(I - D^T)$ for D compact.
(another example of compact ops behaving like finite-dim matrices). (Thm 4.15 Kress, Lin. Int. Eqns).

So $D + \frac{1}{2}$ is not invertible when $k = k_j$

$\Rightarrow$ Our double-layer BIE for scattering, $(D + \frac{1}{2})\tau = -u^i$, <u>fails</u> at $k = k_j$.

The fix:

'Mix an imaginary amount of single-layer in'

Repr, $u = (D - i\eta S)\tau$

choose $\eta > 0$
optimal, Kress suggests $\eta = k$.

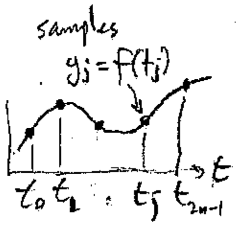
BIE becomes $\underbrace{(D - i\eta S + \tfrac{1}{2})}_{\text{can prove this is not singular for any real } k > 0}\tau = -u^i$ (next time)

Recall $S(x,y)$ has log singularity, <u>so</u> to get <u>accurate</u> Nyström method, need <u>special</u> quadrature.

QUADRATURE RULES for LOG SINGULARITY ... a start.

Interlude on <u>Interpolation</u>.

Given cont. func. f, approximate by $\displaystyle\sum_{k=0}^{2n-1} a_k u_k =: f_n \in \text{Span}\{u_k\}$ ← basis functions, $2n$ of them.



We want f_n to match f at $2n$ collocation points $\{t_j\}_{j=0\cdots 2n-1}$

ie $\displaystyle\sum_{k=0}^{2n-1} a_k u_k(t_j) = y_j \qquad j = 0 \cdots 2n-1$

If matrix $u_k(t_j)$ nonsingular then $\{a_k\}$ unique for any set $\{y_j\}$

f_n is then a <u>reconstruction</u> of f using just the values at colloc. pts.
(ie interpolation.)

There are many possible sets of u_k, eg

- polynomials t^k
- piecewise polynomials (splines)
- trigonometric polynomials $\left\{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right\} kt$

sets of t_j, eg. uniformly spaced, 'graded mesh', etc.

Eg. trig. polynomial on $[0, 2\pi]$ periodic funcs:

$$f_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n-1}\left(a_k \cos kt + b_k \sin kt\right) + \frac{a_n}{2}\cos nt$$

Choose $t_j = \frac{j\pi}{n}$, $j = 0 \cdots 2n-1$, uniformly spaced. (Fourier series)

<u>Analytic</u> Formulae for coeffs given y_j function samples:

$$\left\{\begin{aligned} a_k &= \frac{1}{n}\sum_{j=0}^{2n-1} y_j \cos kt_j \\ b_k &= \frac{1}{n}\sum_{j=0}^{2n-1} y_j \sin kt_j \end{aligned}\right.$$

Why? Fourier inversion

start with
$$\sum_{k=0}^{2n-1} e^{i\pi \frac{jk}{n}} = \begin{cases} \sum_k 1 & \text{for } j=0 \\ \dfrac{1-(e^{i\pi j/n})^{2n}}{1-e^{i\pi j/n}} & j \neq 0 \end{cases} = 2n\,\delta_{jk}$$

Kronecker delta.

↖ geom. sum, numerator vanishes.

Math 116 LECTURE 9

Thurs. 2/2/06 (1)
Barnett.

Principles of successful coding (Alex's tips):

- sit down away from computer & decide in what order things get done. Draw flowchart, etc: setup bdry → fill matrices → solve $Ax=b$ → plot answer.
- write modular code. Modules are blocks of code which talk to each other minimally & perform a defined task.
  - eg. functions / subroutines ... useful since can call repeatedly (in a loop).
  - before you code, think about the interface. Eg. the way we set up dipole.m in HW1 had well-considered inputs & outputs.
  - make code (modules) reflect the mathematics. Eg dipole.m corresponded to one equation from the theory, but knew nothing about N, the shape, etc.
  - Put all 'user' parameters at top of code, and make everything depend on them. Eg. N=50; should be set once, trickles everywhere. Eg. $f(\theta) = 1 + \underbrace{0.3}_{\text{or call 'a' for generality}} \cos 3\theta$ should be defined once.
  - Test each step as you go: be creative in devising a test with a known answer. Observing that there's no crash is not a test! Eg. set $a=0$, gives a circle, which you can solve analytically.

This is essence of object-oriented programming. (You can do it in any language, not just C++, java)

Why bother? It's exponentially easier to debug, change, reuse, generalize, document...

- Think about making an easy-to-use package (document each routine, preferably.) for the 'user' (you, your future self, or others!)
- Look at other code examples (websites, tutorials, books, classmates/peers)

  eg. "Spectral Methods in MATLAB", L.N. Trefethen book

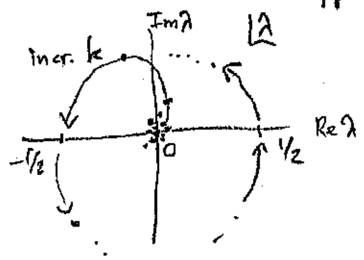
  "Intro to PDE with MATLAB", J. Cooper book.
- Plot everything to check it ~ & beautiful plots attract attention!

[More on interior resonance fix:] [Context: double-layer method for wave scattering] (2)

Recall BIE $\underbrace{(D+\tfrac{1}{2})}_{\text{goes singular}}\tau = f$ fails as $k\to k_j$, since $(D^T+\tfrac{1}{2})\sigma = 0$ defines interior Neumann eigenvalues $\{k_j\}$

Let's watch this happen:

- eigenvalues λ_ℓ of D in $\mathbb{C}$ emerge from origin and 'hit' $-\tfrac{1}{2}$ as wavenumber k increased.



- After hitting they sail around and condense on circle radius $\tfrac{1}{2}$.

This means $2D$ is approximately unitary in some subspace of dimension $\propto N(k) := \#\{j : k_j < k\}$

Why? Project idea. (semiclassical).

we will learn as $k\to\infty$ this scales like volume$(\Omega)\cdot k^d$

- Note each λ also passes through $+\tfrac{1}{2}$!

Why? Interior Dirichlet eigenmodes $\begin{cases}(\Delta + k_j^{(D)2})u_j = 0 & \text{in } \Omega \\ u_j = 0 & \text{on } \partial\Omega\end{cases}$

JR4: limiting value on $\partial\Omega$, approaching from inside is $u^- = (D-\tfrac{1}{2})\tau$ if u rep. by double-layer potential. So eigenmodes have $(D-\tfrac{1}{2})\tau = 0$

Therefore a $\lambda\to+\tfrac{1}{2}$ when $k\to k_j^{(D)}$

This is a popular way to find eigenmodes. Try it! (HW3).

The fix: use representation $u(x) = ((D - i\eta S)\tau)(x)$

x outside Ω. η = some const >0.

JR4 gives BIE $(D - i\eta S + \tfrac{1}{2})\tau = f$ Brakhage–Werner, Leis, Panich (1960's).

Why never singular? (see Colton–Kress "Inverse..." book, p. 48–49, 2nd Ed.).

Suppose $(D - i\eta S + \tfrac{1}{2})\tau = 0$ We wish to show $\tau\equiv 0$ follows, ie $D + i\eta S + \tfrac{1}{2}$ is injective.

so $u = (D - i\eta S)\tau$ has $u^+ = 0$ by construction of BIE.

$\Rightarrow$ $u = 0$ in all of $\mathbb{R}^d\setminus\overline{\Omega}$ outside!, by uniqueness of exterior Dirichlet problem.

$\Rightarrow$ $u_n^+ = 0$ too.

Use jumps in u, u_n to get inside:

JR1,4 $\Rightarrow$ $u^- = -\tau$

JR2,3 $\Rightarrow$ $u_n^- = -i\eta\tau$

GT1 applied inside Ω gives $\int_{\partial\Omega} \bar{u}\, u_n\, ds = \int_\Omega u\Delta\bar{u} + \vec{\nabla}\bar{u}\cdot\vec{\nabla}u\, dx$ (3)

from above

$i\eta \int_{\partial\Omega} |\tau|^2 ds$ $\qquad$ $\underbrace{\int_\Omega -k|u|^2 + |\vec{\nabla}u|^2\, dx}_{\text{pure real}}$

Take Im part of eqn shows $\tau \equiv 0$. QED

Essentially we have shown $\eta \int_{\partial\Omega} |\tau|^2 ds$ is <u>flux</u> entering domain Ω, but this vanishes for a Helmholtz soln. everywhere in domain.

Remarks

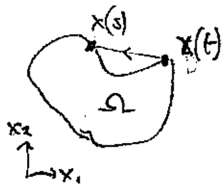
- This is both an analytic tool (to prove existence/uniqueness of scattering solutions) and <u>numerical</u>.
- I believe, sign of η immaterial for numerical purposes.
- Watch fixed eigenvalues λ of $D - i\eta S$ move ... they avoid $-\frac{1}{2}$ like crazy.

$\boxed{\text{Modifying Nyström method for } D - i\eta S}$: $\qquad d=2$

Recall $\quad \Phi(x,y) = \frac{i}{4} H_0(k|x-y|) \sim \underbrace{\frac{1}{2\pi}\ln\frac{1}{|x-y|}}_{\text{singular}} + O(1)$

I omit (1)

this is same as weight func you used $w_j = |\dot{x}|\, d\theta$



parametrize $\partial\Omega$ by $x(t)$, $t \in [0, 2\pi)$ $\qquad ds_y = |\dot{x}(t)|\, dt$

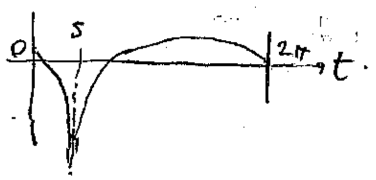
Then $(S\tau)(s) = \int_{\partial\Omega} \Phi(x,y)\,\tau(y)\, ds_y = \int_0^{2\pi} \underbrace{\frac{i}{4} H_0(k|x(s)-x(t)|)\,|\dot{x}(t)|}_{=: M(s,t)}\, dt$

We want to write $M(s,t) = M_1(s,t)\cdot(\ln|s-t| \text{ singular}) + M_2(s,t)$

with both M_1, M_2 analytic, and our 'ln singular' function $\begin{cases}\text{periodic}\\ \text{easy to analyse}\end{cases}$

We choose $\ln\left(4\sin^2\frac{s-t}{2}\right)$ as periodic ln-singular function .... it will have <u>known</u> Fourier coeffs.

$\sim 2\ln|s-t|$ as $s-t \to 0$, ie 'strength' is 2.



note $\frac{s-t}{2}$ so only has 1 singularity per period.

Have $M(s,t) = \overbrace{\frac{-1}{4\pi}\underbrace{J_0(k|x(s)-x(t)|)}_{\text{limit of 1 on diag.}}|\dot{x}(t)|}^{M_1(s,t)} \cdot \ln\left(4\sin^2\frac{s-t}{2}\right) + M_2(s,t)$ $\qquad (*)$

$M_2(s,t)$ has no singularity as $s \to t$, is analytic, and has $M_2(s,s) = \left[\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{4\pi}\ln\left(\frac{k^2|\dot{x}|^2}{4}\right)\right]|\dot{x}|$

(7)

Here $C = \lim_{p\to\infty} \left\{ \sum_{m=1}^{p} \frac{1}{m} - \ln p \right\} = 0.57\cdots$ is Euler's const.

Note you now can complete M_1 & M_2 at any s,t (use (*) for M_2)

We may split up $\frac{\partial \Phi}{\partial n_y}(x,y)$ for $\mathbb{D}\tau(s)$ in similar way [see Kress review].

Thus our BIE is
$$\int_0^{2\pi} K(s,t)\,\tau(t)\,dt + \tfrac{1}{2}\tau(s) = f(s)$$

with
$$K(s,t) = \underbrace{K_1(s,t)}_{\text{analytic}} \ln\left(4\sin^2\frac{s-t}{2}\right) + \underbrace{K_2(s,t)}_{\text{analytic}}$$

$\boxed{\text{Quadrature:}}$

Using uniform quadrature in t variable is what you already do (t = angle variable so weights $w_j = |\dot{x}(t_j)|\Delta\theta$)
This gave exponential convergence for analytic kernels (eg. $D(s,t)$).
Beautiful thing: can get exponential ('spectral') convergence also for above log singularity!

Take $2n$ equally-spaced quadrature points t_j: $j=0, 1, \ldots, 2n-1$ on $[0, 2\pi)$, $t_j = \frac{\pi j}{n}$

Analytic integrand
$$\int_0^{2\pi} K_2(s,t)\,\tau(t)\,dt \approx \underbrace{\frac{\pi}{n}}_{\text{all weights constant}} \sum_{j=0}^{2n-1} K_2(s,t_j)\,\tau(t_j)$$

Log sing. analytic
$$\int_0^{2\pi} K_1(s,t)\ln\left(4\sin^2\frac{s-t}{2}\right)\tau(t)\,dt \approx 2\pi\sum_{j=0}^{2n-1} \underbrace{R_j^{(n)}(s)}_{s\text{-dep. weights}} K_1(s,t_j)\,\tau(t_j)$$

translational invariance, $R_j^{(n)}(s) = R_0^{(n)}(s-s_j)$

$\Rightarrow$ Nyström method will be, by setting $s = t_i$:
$$\sum_{j=0}^{2n-1} \underbrace{\left[\frac{\pi}{n} K_2(t_i,t_j) + 2\pi R_0^{(n)}(s_i - s_j) K_1(t_i,t_j)\right]}_{\text{this is your new "}\tilde{K}\text{" matrix}} \tau(t_j) - \tau(t_i) = f(t_i)$$

Note $R_0^{(n)}(s_i - s_j) = R_{|i-j|}^{(n)}(0)$ Let's now get $R_j^{(n)}(s)$ ...

Barnett

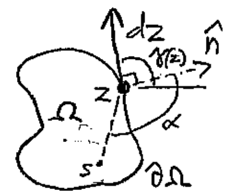
Math 116 — LECTURE 10

Today we show:

- Complex contour integration intimately related to Laplace double layer
- $\ln\left(4\sin^2\frac{s-t}{2}\right)$ is kernel of 'Neumann-to-Dirichlet' map on unit disk
- How to interpolate periodic funcs with Fourier series
- derive spectral (exponentially convergent) log-singularity quadrature.

Cauchy contour integral:

$\partial\Omega$ closed curve in $\mathbb{C} \equiv \mathbb{R}^2$



$$\frac{1}{2\pi i}\oint_{\partial\Omega}\frac{f(z)}{z-s}\,dz = \begin{cases} f(s) & \text{if } s\in\Omega \\ 0 & \text{if } s\in\mathbb{C}\setminus\overline{\Omega}\end{cases}$$

for f analytic in $\overline{\Omega}$

complex contour integral converts to line integral via $dz = e^{i\gamma(z)}ds$

tangent angle at z. arclength

$$\mathrm{Re}\,\underbrace{\frac{e^{i\gamma(z)}}{i}}_{\hat{n}}\cdot\frac{1}{z-s} = -\frac{\cos\alpha(z,s)}{|z-s|}$$

α is angle of $(z-s)$ to $\hat{n}$. Laplace $\partial\Phi(s,z)/\partial n_z$

So for real f, real part of above Cauchy integral $= -\int_{\partial\Omega}\overbrace{\frac{\cos\alpha(z,s)}{2\pi|z-s|}}\,f(z)\,ds$

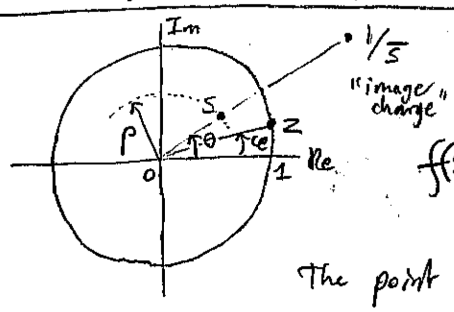
$= -(\mathcal{D}f)(s)$ double-layer!

Remarks

1) imaginary part of Cauchy integral is (strongly) singular,
thus complex analysis allows proof of results on singular integral equations (Kress "Linear I.Es.", ch.7)

2) Cauchy Theorem looks like "$f(s) = (\mathcal{D}f)(s)$", ie tempting to think in the real case surface density is just $\tau = -f|_{\partial\Omega}$, and $f\equiv 0$ for s outside. These are wrong since $\mathcal{D}$ does not include Im part. Beauty of complex case is that τ doesn't need to be solved for!

Unit disk : Poisson kernel



An example of Cauchy integral, write $z = e^{i\varphi}$, $s = \rho e^{i\theta}$, $dz = iz\,d\varphi$

to line (angle) integral

$$f(s) = \frac{1}{2\pi i}\oint\frac{f(z)}{z-s}\,dz = \frac{1}{2\pi}\int_0^{2\pi}\frac{z}{z-s}f(z)\,d\varphi \quad \text{for } s \text{ inside}$$

The point $1/\bar{s}$ is outside, so $0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{z}{z-1/\bar{s}}f(z)\,d\varphi$ also holds.

Subtract the two equations: $\longrightarrow$ convert to $\frac{-\bar{s}}{\bar{z}-\bar{s}}$ using $z\bar{z}=1$

$$f(s) = \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{z}{z-s} - \frac{z}{z-1/\bar{s}}\right] f(z)\, d\varphi$$

$\longrightarrow$ factorize $\quad \frac{z(\bar{z}-\cancel{\bar{s}}) + \bar{s}(\cancel{z}-s)}{(z-s)(\bar{z}-\bar{s})} = \frac{1-|s|^2}{|z-s|^2}$

$$= \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\varphi)} \quad \swarrow \text{cosine rule}$$

So Poisson kernel representation of interior values of f is

$$f(\rho,\theta) = f(s) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\varphi)}\, f(\varphi)\, d\varphi$$

$\swarrow$ boundary values.

- Notice this is more 'user-friendly' than GRF since only boundary values (not derivs) needed.
- It solves Dirichlet interior BVP directly.
- The kernel $\frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\phi)}$, sometimes called 'harmonic measure', is $\frac{\partial G(x,y)}{\partial n_y}$ where G is Green's function <u>for the domain</u>. It is <u>not</u> same as kernel of layer potentials $\mathcal{D}, \mathcal{S}$, etc.
  $\nwarrow$ in the unit disk G analytically known (via image charge $\frac{1}{\bar{s}}$)

$\boxed{\text{Neumann-to-Dirichlet map}}$:

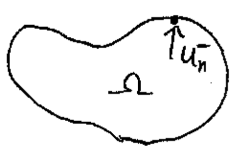
u harmonic in Ω

given $u_n|_{\partial\Omega}$, what is $u|_{\partial\Omega}$? Map: $u = Au_n$

eg injected current density into surface

eg measured voltages on surface

$\underbrace{\qquad\qquad\qquad\qquad}$ Modern (& medically relevant) imaging tool: Electrical Impedance Tomography (EIT).



A can be written in terms of layer potentials.

Recall 'zero flux' corollary $\int_{\partial\Omega} u_n\, ds = 0$ for harmonic funcs.

$\Rightarrow$ domain of A is the functions $C_0(\partial\Omega)$, <u>zero mean on $\partial\Omega$</u> (otherwise no such u exists).

Recall we may add a const to u without changing u_n, so Au_n unique only up to const.

Say $u_n = g$, given Neumann data, represent u inside by $u(x) = (\mathcal{S}\sigma)(x)$

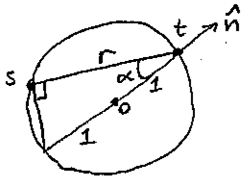
JR2: $\quad g = u_n^- = (\mathcal{D}^T + \tfrac{1}{2})\sigma$

so $\quad \sigma = (\mathcal{D}^T + 1/2)^{-1} g \quad \longleftarrow$ since we've restricted to $C_0(\partial\Omega)$, inverse exists, bounded.

JR1: $\quad u^- = \mathcal{S}\sigma$

Combining: $\quad u|_{\partial\Omega} = \underbrace{\mathcal{S}(\mathcal{D}^T + 1/2)^{-1}}_{\text{this is } A} g$

$\boxed{\text{ND map for unit disk:}}$



recall
$$D(s,t) = -\frac{1}{2\pi}\underbrace{\frac{\cos\alpha}{r}}_{\equiv \frac{1}{2}}$$
$\equiv \frac{1}{2}$, see triangle which is right by geom.
$$= \frac{-1}{4\pi}, \quad \forall s,t.$$

This matches $D(s,s) = -\frac{\kappa(s)}{4\pi}$ since $\kappa \equiv 1$ for unit disk.

since $D(s,t)$ const,

(JR2) $$g(s) = \int_0^{2\pi} \frac{-1}{4\pi}\sigma(t)\,dt + \frac{1}{2}\sigma(s) = \frac{1}{2}\left(\sigma(s) - \langle\sigma\rangle\right)$$

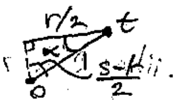
mean value of σ (undetermined by g)

So $(D^T + 1/2)^{-1} : C_0(\partial\Omega) \to C_0(\partial\Omega)$ is 1-to-1, and is just $2I$

So ND map is $A = S(D^T + 1/2)^{-1} = 2S$ the single-layer operator.

In other words $u(s) = \int_0^{2\pi} A(s,t)\, g(t)\,dt$

using triangle



with kernel $$A(s,t) = -\frac{1}{\pi}\ln r = -\frac{1}{\pi}\ln\left(2\sin\frac{s-t}{2}\right)$$
$$= \underline{-\frac{1}{2\pi}\ln\left(4\sin^2\frac{s-t}{2}\right)}$$

- We can evaluate its Fourier coeffs using complex monomials $\mathrm{Re}(z^m) = \rho^m e^{im\theta} =: u^{(m)}$

$g(t) = \left.\frac{\partial u^{(m)}}{\partial n}\right|_{\rho=1} = m e^{imt}$, Apply ND map gives, for $m>0$, boundary values $u^{(m)}|_{\rho=1} = e^{imt}$

$$\Rightarrow \quad e^{ims} = -\frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\frac{s-t}{2}\right)\cdot m e^{imt}\,dt \quad , \quad m>0$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_0^{2\pi}\ln\left(4\sin^2\frac{s-t}{2}\right)e^{imt}\,dt = \begin{cases} 0 & m=0 \\ \dfrac{-e^{ims}}{|m|} & m\neq 0 \end{cases}$$

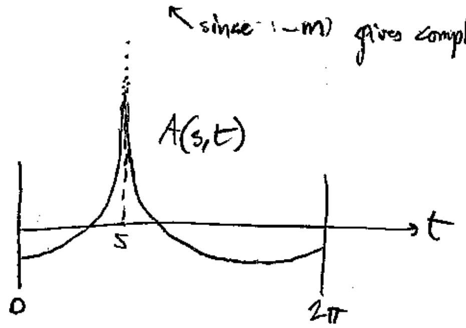
Fourier coeffs (Lemma 8.21, Kress, "Lin. Int. Eq.")

$\leftarrow$ since range of ND map is zero-mean funcs, $C_0(\partial\Omega)$

since $-m$ gives complex conj. of $+m$.

- 'Unwrapping' single layer source

gives plot



amazingly this singularity (which is in $L^2[0,2\pi]$) has Fourier coeffs dying like $O(\frac{1}{m})$, same as jump discontinuity.

Interpolation of functions:

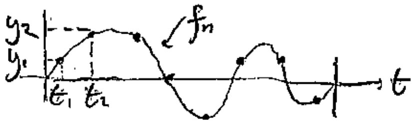
[We'll also build tools to do spectrally accurate integration of a (log singularity) × (analytic func.)]

Goal: given samples of f at points t_j, $j = 0 \ldots 2n-1$, reconstruct smooth approx to f everywhere.

Our approximation to f, called f_n, will lie in $X_n := \text{Span}\{u_k\}$, $k = 0 \cdots 2n-1$

Samples are $y_j := f(t_j)$ ↳ ie $f_n(t) = \sum_k a_k u_k(t)$ ← basis funcs.

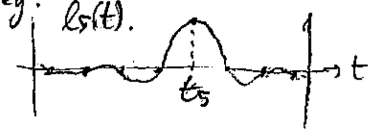
- If matrix with kj entry $u_k(t_j)$ is nonsingular, there is unique element f_n of $\text{Span}\{u_k\}$ which matches y_j at t_j, $\forall j$



- Linear map $L_n f = f_n$, is a projection since $L_n^2 = L_n$, since $L_n f$ already matches at points.

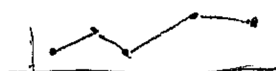
- There is a unique element l_k of X_n, for each k, for which $l_k(t_j) = \delta_{jk}$ Kronecker delta. $\forall j$

eg. $l_5(t)$.

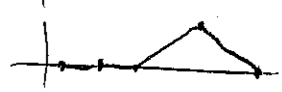


called 'Lagrange polynomial'.

eg. X_n = piecewise linear between points x_j (splines):

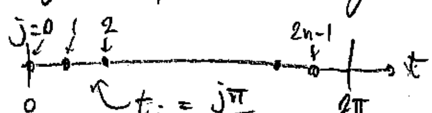


- Note $L_n f = \sum_{j=0}^{2n-1} y_j l_j$ ← check it matches!



, all l_k's are triangular hat funcs.

- $\boxed{\text{Key example}}$: 'trigonometric polynomials' on uniform grids.

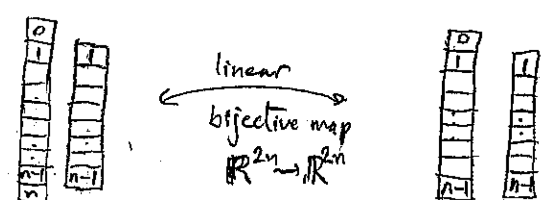


$$f_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(a_k \cos kt + b_k \sin kt) + \frac{a_n}{2}\cos nt$$

$f_n \in T_n$, n^{th}-order trig polys, is actually of dim $2n+1$, but $\sin nt$ dropped since vanishes at all t_j.

$$= \frac{\hat{f}_{-n}}{2}e^{-int} + \sum_{k=-n+1}^{n-1}\hat{f}_k e^{ikt} + \frac{\hat{f}_n}{2}e^{int} =: \sum_k{}' \hat{f}_k e^{ikt}$$

Fourier series representation.
Note $k = \pm n$ terms really contribute as one; $\hat{f}_n = \hat{f}_{-n}$



a_k b_k
$2n$ real params

$\text{Re}\hat{f}_k$ $\text{Im}\hat{f}_k$
also $2n$ real params.

$\hat{f}_{-k} = \overline{\hat{f}_k}$ since f real.

BACKWARD MAP

$$\begin{aligned} a_0 &= 2\hat{f}_0 \\ a_k &= \hat{f}_k + \hat{f}_{-k} \\ b_k &= i\hat{f}_k - i\hat{f}_{-k} \\ a_n &= 2\hat{f}_n = 2\hat{f}_{-n} \end{aligned} \quad \Big\} \; k = 1 \cdots n-1$$

FORWARD MAP

$$\left.\begin{aligned} \hat{f}_k &= \tfrac{1}{2}(a_k - ib_k) \\ \hat{f}_{-k} &= \tfrac{1}{2}(a_k + ib_k) \end{aligned}\right\} k = 0 \cdots n$$

defining $b_n = b_0 = 0$

Miraculous sum of exp: $k \in \mathbb{Z}$, $\displaystyle\sum_{j=0}^{2n-1} e^{i\pi\frac{jk}{n}} = 2n\,\delta_{k,0}$ where $\delta_{k,m} := \begin{cases} 1, & k \equiv m \pmod{2n} \\ 0, & \text{otherwise} \end{cases}$

Gives useful formulae $\displaystyle\sum_{j=0}^{2n-1} e^{ikt_j} = 2n\,\delta_{k,0}$ sum over grid pts

$\displaystyle{\sum_k}' e^{ikt_j} = 2n\,\delta_{j,0}$ sum over frequencies

• Finding interpolant means getting $\{\hat{f}_k\}$ from $\{y_j\}$, such that $y_j = f_n(t_j)$ $j = 0 \cdots 2n-1$

i.e. $\displaystyle y_j = {\sum_k}' \hat{f}_k e^{ikt_j}$

mult by e^{-imt_j} & sum j

$$\sum_{j=0}^{2n-1} y_j e^{-imt_j} = {\sum_k}' \hat{f}_k \underbrace{\sum_{j=0}^{2n-1} e^{i(k-m)t_j}}_{2n\,\delta_{k,m}} = 2n\,\hat{f}_m \quad \text{for } m = -n, \dots, n.$$

(note $m = \pm n$ case used fact $\hat{f}_n = \hat{f}_{-n}$)

Inversion formula $\boxed{\hat{f}_m = \frac{1}{2n}\sum_{j=0}^{2n-1} y_j e^{-imt_j}}$ for $m = -n \dots n$.

so coeffs unique given $\{y_j\}$

Note matrix $A_{jk} := \frac{1}{\sqrt{2n}} e^{ikt_j}$ unitary.

• Lagrange poly: ℓ_k has mth fourier coeff $\displaystyle\frac{1}{2n}\sum_{j=0}^{2n-1} \delta_{jk} e^{-imt_j} = \frac{e^{-imt_k}}{2n}$ (desired 'y_j' for lagrange.)

$$\Rightarrow \quad \ell_k(t) = {\sum_m}' \frac{e^{-imt_k}}{2n} e^{imt} = \frac{1}{2n}{\sum_m}' e^{im(t-t_k)} \qquad (*)$$

$$= \frac{1}{2n}\left[1 + 2\sum_{m=1}^{n-1} \cos m(t-t_k) + \cos n(t-t_k)\right] \quad \text{explicitly real expression}$$

$$= \frac{1}{2n} \cot\left(\frac{t-t_k}{2}\right) \sin n(t-t_k) \qquad \text{check in HW3!}$$

Now armed with all Lagrange poly's you build trig. interpolant $\displaystyle f_n(t) = \sum_{k=0}^{2n-1} y_k \ell_k(t)$

Spectral Quadrature Weights:

after all that, Lagrange poly's immediately give weights w_j

Eg. $$\int_0^{2\pi} f(t)\,dt \approx \int_0^{2\pi} f_n(t)\,dt = \sum_{k=0}^{2n-1} y_k \underbrace{\int_0^{2\pi} l_k(t)\,dt}_{\frac{1}{2n}\cdot 2\pi = \frac{\pi}{n}}$$ using (*) and $\int_0^{2\pi} e^{imt}\,dt = \begin{cases} 1, & m=0 \\ 0, & \text{otherwise} \end{cases}$

$$= \sum_{k=0}^{2n-1} w_k f(t_k) \qquad \text{with weights } w_k = \frac{\pi}{n} \quad \forall k.$$

Our rule is exact for $f \in T_n$; if not then error is bounded by interpolation error $\|f - f_n\|_\infty$, exponentially small vs n for analytic $f(t)$.

Eg. $$\int_0^{2\pi} f(t) \ln\left(4\sin^2\frac{s-t}{2}\right) dt \approx \int_0^{2\pi} f_n(t) \ln\left(4\sin^2\frac{s-t}{2}\right) dt = \sum_{k=0}^{2n-1} y_k \underbrace{\int_0^{2\pi} l_k(t) \ln\left(4\sin^2\frac{s-t}{2}\right) dt}_{\text{defines weight } R_k^{(n)}(s)}$$

Using (*),

$$R_k^{(n)}(s) = \frac{1}{2n} \sum_m{}' e^{-imt_k} \underbrace{\int_0^{2\pi} e^{imt} \ln\left(4\sin^2\frac{s-t}{2}\right) dt}_{\begin{cases} 0 & m=0, \text{ by page (3)} \\ -\frac{2\pi e^{ims}}{|m|} & m \neq 0 \end{cases}}$$

$$= -\frac{2\pi}{2n} \sum_{m\neq 0}{}' \frac{1}{|m|} e^{im(s-t_k)}$$

$$\boxed{R_k^{(n)}(s) = -\frac{\pi}{n}\left[2\sum_{m=1}^{n-1} \frac{1}{m}\cos m(s-t_k) + \frac{1}{n}\cos n(s-t_k) \right]}$$

log singularity weights. explicitly real

As above, have derived weights which are exact for $f \in T_n$, exponentially convergent for $f(t)$ analytic. This is how Kress (following Martensen & Kussmaul in 60's) formula comes about (eg. K's review Eqn. (3.1)).