M116 Lec 7: ① Th. 10/16/08

Last time: we can construct $(n+1)$-node Gaussian quadratures if can find $n+1$-degree poly q

i) orthog. to all $\mathbb{P}_n$, in our inner prod $(\cdot,\cdot)$. $q_{n+1}(x) = \prod_{j=0}^{n}(x-x_j)$

ii) with all roots $x_j \in [a,b]$, which give the nodes

Lemma 9.15 $\exists$ unique seq. $(q_n)_{n=0}^{\infty}$ w/ $q_0 = 1$ & $q_n(x) = x^n + p(x)$, $p \in \mathbb{P}_{n-1}$

which are mutually orthog. $q_n \perp q_m$, $n \neq m$, and $\mathrm{Span}\{q_0, \cdots q_n\} = \mathbb{P}_n$

Pf: $1, x, x^2, \cdots$ are Lin. Indep., so Gram-Schmidt unique:

$$q_0 = 1$$
$$q_1 = x - \frac{(x, q_0)}{(q_0, q_0)}$$
$$q_2 = x^2 - \frac{(x^2, q_1)}{(q_1, q_1)} - \frac{(x^2, q_0)}{(q_0, q_0)}$$
$$\vdots$$
$$q_n = x^n - \sum_{j=0}^{n-1}\frac{(x^n, q_j)}{(q_j, q_j)}$$

n L.I. vecs. in $\mathbb{P}_n$ must span it.

'Legendre' poly's (but: not std normalization): unique (unweighted) orthog poly's on $[-1,1]$

Lemma 9.16 q_n has n simple zeros in (a,b).

Pf: $\forall n \geq 1$, $q_n \perp q_0$ ie $\int q_n = 0$ so q_n has ≥ 1 zeros $x_1 \cdots x_m$ in (a,b).

Suppose $m < n$, then $r_m := \prod_{j=1}^{m}(x-x_j) \in \mathbb{P}_{n-1}$ so is $\perp q_n$

But $\int r_m q_n \neq 0$ since $r_m q_n$ has fixed sign, not $\equiv 0$.

} contradicts. $\Rightarrow m = n$.

In practise, how do we compute nodes $\{x_j\}_{j=0}^{n}$? They are eigvals of

$$\begin{bmatrix} 0 & \beta_1 & & & \\ \beta_1 & 0 & \beta_2 & & \\ & \beta_2 & 0 & \beta_3 & \\ & & \beta_3 & 0 & \ddots \\ & & & \ddots & \beta_{n-1} \end{bmatrix}$$

(tridiagonal since 3-term recurrence)

$$\beta_n := \tfrac{1}{2}\left(1-(2n)^{-2}\right)^{-1/2}$$

& weights w_j come from 1st component of eigenvectors. [Wilf]

See gauss.m code., Golub-Welsch scheme.

We could prove but won't. (beautiful)

Claim: $2n+1$ is highest possible degree of $(n+1)$-node quadrature.

pf: $p = \prod_{j=0}^{n}(x-x_j)^2 \in \mathbb{P}_{2n+2}$ has $Q_n(p) = 0$ but $Q(p) > 0$.

Thm: Gaussian weights non-neg: pf $l_k(x_j) = \delta_{jk}$ so $l_k^2(x_j) = \delta_{jk}$ also.

$$\text{so } 0 < \underbrace{\int l_k^2(x)\,dx}_{\text{nonneg}} \underset{\substack{\text{exact since} \\ l_k^2 \in \mathbb{P}_{2n}}}{=} \sum_{j=0}^{n} w_j\, l_k^2(x_j) = w_k$$

Cor: Gauss. quad convergent.

[There are error bnds for Gauss. quadr., won't do].

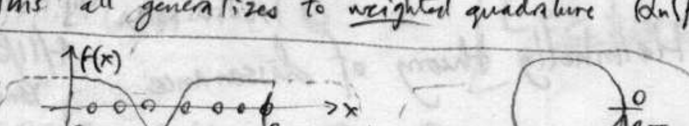
- This all generalizes to weighted quadrature $Q_n(f) := \int_{-1}^{1} f(x)\,\omega(x)\,dx$, in which case weight (we did $\omega \equiv 1$)

inner prod $(f,g) = \int_{-1}^{1} f(x) g(x) \omega(x) dx$. Useful for f with singularities

eg $f(x) = g(x) \ln|x|$ — smooth; move into $\omega(x)$

② 10/16/0

Periodic Quadr:



Periodic trapezoid rule : $Q_n(f) = \frac{2\pi}{n}\sum_{j=1}^{n} f\left(\frac{2\pi j}{n}\right)$ ← x_j equally spaced.

w_j all equal

Could derive via interpolation of f by trigonometric poly's, ie Fourier series truncated at term $n/2$ ... later.

For now : some error bounds, error $E_n(f) := Q_n(f) - Q(f)$, a number.

Thm (9.27) Let $f: \mathbb{R} \to \mathbb{R}$ be 2π-periodic & C^{2m+1}, $m \in \mathbb{N}$,

then $|E_n(f)| \le \underbrace{C_m \int_0^{2\pi} |f^{(2m+1)}(x)|\,dx}_{\text{const dep on } m} \cdot \frac{1}{n^{2m+1}}$ ← convergence.

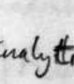
- means if $f \in C^5$, ie $f^{(5)}$ cont. but $f^{(6)}$ discont., quadr. error is $O(n^{-5})$ — 5^{th} order with effort (n = # func evals; $n \to 10n$ gets you 10^5, ie 5 extra digits. But to get 15 extra digits need $10^3 n$)

So, smoother f gives higher-order algebraic convergence.

- Euler-Maclaurin expansion.

What is the smoothest type of func? analytic.

Review complex analysis:



$f(z)$ $z \in \mathbb{C}$ isolated poles ie $b(z-a)^{-k}$ locally

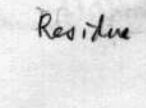
$k=1$: simple pole: $\frac{b}{z-a}$ residue.

$f(z)$ 'holomorphic' in domain $D \subset \mathbb{C}$; no poles inside D, Taylor series converges in some disc.

eg $\frac{1}{1+25x^2}$ has simple poles at $\pm i/5$.

Residue thm: if f holomorphic in D apart from finite # poles,

then $\int_{\partial D} f(z)\,dz = 2\pi i \sum_{\text{simple poles}} (\text{residue of each simple pole})$



line integral.

Why? i) Cauchy f holom $\Rightarrow \int_{\partial D} f(z) dz = 0$

ii) small loops around pole cancels unless $f(z)$ goes CW once (since dz goes CCW once)

Skip to strip theorem & proof.

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- Note $\cot\frac{nz}{2} \to \begin{cases} -1 & \text{for } \mathrm{Im}\,z > 0 \\ +1 & < 0 \end{cases}$ as $n \to \infty$, exponentially fast. show pic.

Note: Gaussian & Newton-Cotes (worse) & periodic trap. examples of 'spectral methods', ie exp. conv, error $O(K^{-n})$ ie e^{-cn}

You may also take derivatives of (trig) polys to get formulae for derivatives f', f'' etc, then solve ODEs (or PDEs) with spectral accuracy.

Math 116. [Lecture 6 from 2006] ① 1/24/06 Barnett

ERROR ANALYSIS of INTEGRATION OF PERIODIC FUNCS.

Why is crude equal-weight equally-spaced quadrature $\int_0^{2\pi} g(x)dx \approx \underbrace{\frac{2\pi}{N}}_{w_j \text{ all equal.}}\sum_{j=1}^{N} g\left(\frac{2\pi j}{N}\right)$ so good?

ANALYTIC CASE (§9.4, Kress, "Numerical Analysis").

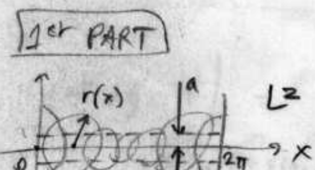
Thm. Let $g: \mathbb{R} \to \mathbb{R}$ be analytic & 2π-periodic. Then there exists a strip $D = \mathbb{R} \times (-a, a) \subset \mathbb{C}$ with $a > 0$ s.t. g can be extended to a holomorphic and 2π-periodic bounded function $g: D \to \mathbb{C}$.
The error for above quadrature rule is bounded by

$$|R_N[g]| \leq \frac{4\pi M}{e^{Na} - 1}$$

where M is a bound for holomorphic function g on D.

Remark: • this proves exponential convergence of errors $O(e^{-aN})$ a vertical dist. to nearest pole of f.
• you'll test in HW.

Proof: $\boxed{1^{st} \text{ PART}}$



Analytic ⇒ at each $x \in \mathbb{R}$, Taylor expansion converges in some open disk radius $r(x) > 0$.
This provides a 2π-periodic holomorphic extension of g. since x & $x+2\pi$ have same Taylor expansion.

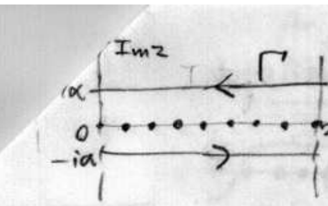
Can cover $[0, 2\pi]$ with finite # of such disks.
a can be chosen to be any width $<$ minimum $r(x)$.
g is then bounded on the strip D.

$\boxed{2^{nd} \text{ PART}}$

Consider $\cot(z)$, which has residuals (pole strengths) of 1 at $z_j = \pi j$, $j \in \mathbb{Z}$ (since $\frac{d}{dz}\Big|_0 \tan z =$

Then $g(z)\cot\left(\frac{N}{2}z\right)$ has residuals $\frac{2}{N} g\left(\frac{2\pi j}{N}\right)$

at points $z_j = \frac{2\pi j}{N}$



Residue thm gives, for $\alpha < a$,

$$\int_\Gamma g(z)\cot\left(\frac{Nz}{2}\right)dz = 2\pi i \sum \text{residuals}$$

$$= \frac{4\pi i}{N}\sum_{j=1}^{N} g\left(\frac{2\pi j}{N}\right) \quad (*)$$

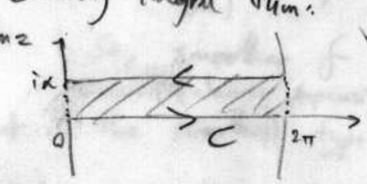
note periodic
⇒ doesn't need to close.

Schwarz reflection principle : $\begin{cases} g \text{ real on } \mathbb{R} \text{ so } g(\bar z) = \overline{g(z)} \\ \text{ie imaginary part is antisymmetric in } \operatorname{Im} z. \end{cases}$

⇒ LHS integral becomes $-i\int_{i\alpha}^{i\alpha+2\pi} 2\operatorname{Im}\, g(z)\cot\left(\frac{Nz}{2}\right)dz = 2i\operatorname{Re}\int_{i\alpha}^{i\alpha+2\pi} i\,g(z)\cot\left(\frac{Nz}{2}\right)dz$

Using (*), $\operatorname{Re}\int_{i\alpha}^{i\alpha+2\pi} i\cot\left(\frac{Nz}{2}\right)g(z)\,dz = \frac{2\pi}{N}\sum_{j=1}^{N} g\left(\frac{2\pi j}{N}\right)$ ~ our quadrature rule!

Cauchy integral thm:



$$\oint_C g(z)\,dz = 0 \quad \text{since analytic in } D$$

so $\operatorname{Re}\int_{i\alpha}^{i\alpha+2\pi} g(z)\,dz = \int_0^{2\pi} g(x)\,dx$ ← true integral

⇒ error $R_N[g] = \operatorname{Re}\int_{i\alpha}^{i\alpha+2\pi}\underbrace{\left[1 - i\cot\left(\frac{Nz}{2}\right)\right]}_{\text{bounded? } y=\frac{Nz}{2}} g(z)\,dz$ — $|g(z)|$ on $x+i\alpha$ is bounded by M.

$$\left|1 - i\cot y\right| \underset{\text{Euler for sin, cos.}}{=} \left|1 + \frac{1+e^{-2iy}}{1-e^{-2iy}}\right| = \left|\frac{2}{e^{-2iy}-1}\right| \le \frac{2}{e^{N\alpha}-1}$$

use $|e^{-2iy}| = e^{2\operatorname{Im} y} = e^{N\alpha}$

Take limit $\alpha \to a$. QED.

Remark : $\frac{1}{\pi N}\operatorname{Im}\cot\frac{Nz}{2}$ is just an approximation to double layer potential placed along the Re axis

≈ -1

$\approx -1.$

cool!

There also exist Euler-Maclaurin theorems for C^{2m+1} FUNCTIONS:

Thm: Let $g \in C^{2m+1}$ be 2π-periodic, for some $m \ge 1$.
Then $|R_N[g]| \le \frac{C}{N^{2m+1}}\int_0^{2\pi}|g^{(2m+1)}(x)|\,dx$ where $C = 2\sum_{k=1}^{\infty}\frac{1}{k^{2m+1}}$

Proof requires Bernoulli poly's (see Kress §9.4).

$\boxed{\text{Smoother } g \Rightarrow \text{higher-order convergence}}$

~~Lec 7~~ Lec 8 :

① 10/22

Integral Eqns.

Given interval $[a,b]$, func f, func of 2 vars k.

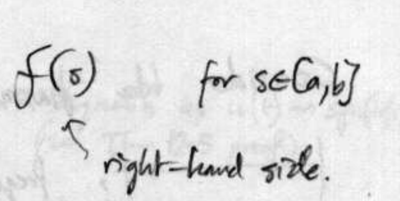
Seek function $u(t)$ solving $\int_a^b k(s,t)\, u(t)\, dt = f(s)$ for $s \in [a,b]$

k: kernel func on the square $[a,b]^2$. f: right-hand side.

'Fredholm eqn of 1st kind'.

Compare linear sys. of eqns: $\sum_j a_{ij} x_j = b_i$, $i = 1 \cdots n$

summarize: $A\vec{x} = \vec{b}$ (A matrix; $\vec{x}$, $\vec{b}$ vectors.)

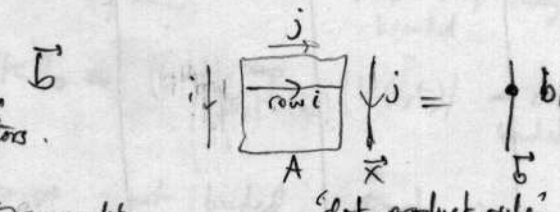


'dot-product rule'.

Summarize IE by $Ku = f$ — K: lin. operator; u, f: funcs; equality as funcs.

defined by $(Ku)(s) = \int_a^b k(s,t)\, u(t)\, dt$

$f(s)$ is $k(s,\cdot)$ dot u — $k(s,\cdot)$: 'row' of kernel.



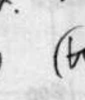
eg. Fourier trans $k(s,t) = e^{ist}$ on $\mathbb{R}$.
convolution; blurring of image $k(s,t) = k(s-t)$ on $\mathbb{R}$

- Lin ops (like square matrices): i) they have eigenvalues? $Ku = \lambda u$ ← eigfunc, even SVD.
  ii) you can apply them repeatedly, eg $(K^2u)(s) = \int K(s,t)(Ku)(t)\,dt = \int K(s,t) \int K(t,r) u(r)\, dr\, dt$
  $= \int \underbrace{k'(s,r)}_{\text{kernel of prod.}} u(r)\, dr$ like an $\int k(s,t) k(\ldots$

Fredholm eqn 2nd kind:
$u - Ku = f$ ie $(I - K)u = f$
(sometimes nonzero const. here.)

- We will use these to solve PDEs. First we solve IEs numerically. This will involve {quadrature, linear systems, func-analysis (matrices finite-d, ∞-dim K brings many twists)}.

Eg. $k(s,t) = \begin{cases} 0 & s \le t \\ 1 & s > t \end{cases}$



'lower-triangular', or 'causal' op.

$(Ku)(s) = \underbrace{\int_a^s u(t)\, dt}_{\text{indef. integral}} = f(s)$ unique solution is $u = f'$, exists iff $f(0) = 0$ & f diff'ble.

This example of Volterra IE: $k = 0$ for $s < t$ (above diag). Easy to solve (equiv to ODE), won't concern us.
Fredholm have stuff on both sides of diag.

Eg. $\int_0^1 \underbrace{s^2 t}_{k(s,t)}\, u(t)\, dt = \frac{1}{3} s^2$ easily solve, bring out s^2: $\int_0^1 t\, u(t)\, dt = \frac{1}{3}$

$u(t) = t + (\text{any func} \perp t)$

K is rank-1 since $(Ku)(s) = c s^2$ — highly nonunique, typical of 1st kind.

Bounded operators:

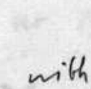
$\|K\| = \sup_{\|u\|=1} \|Ku\|$ for norms of your choice on functions, eg. $L^\infty[a,b]$ norm (sup norm) $\|u\|_\infty := \max_{s \in [a,b]} |u(s)|$

② 10/24

What is op. norm in terms of kernel?

$|(Ku)(s)| = \left|\int_a^b k(s,t)\,u(t)\,dt\right| \le \int_a^b |k(s,t)|\,dt$ if $\|u\|_\infty = 1$. , with equality as $u(t) \to \mathrm{sgn}(k(s,t))$ (see Thm 12.5 proof)

$\|K\|_\infty = \sup_{s\in[a,b]} |(Ku)(s)| = \sup_{s\in[a,b]} \int_a^b |k(s,t)|\,dt$: biggest row-integral of abs val of kernel



Say $b-a<\infty$ and $k \in C([a,b]^2)$ cont. on square, then $\|K\|_\infty \le (b-a) \sup_{s,t\in[a,b]} |k(s,t)| < \infty$ since cont, always bounded. bnded

But $k(s,t) = \dfrac{1}{|s-t|^\gamma}$, $\gamma = 1$: $\int |k(s,t)|\,dt \to \infty$ not bnded, strongly singular.

$0 < \gamma < 1$ integrable $\to$ bnded, 'weakly-singular' kernel. (blows up, discontinuous at $s=t$)

May instead use L^2 norm: $\|u\|_2 := \sqrt{\int_a^b |u(s)|^2 ds}$, Eg $|(Ku)(s)| \le \left|\int k(s,t)u(t)\,dt\right| \overset{C-S}{\le} \underbrace{\sqrt{\int |k(s,t)|^2 dt}}_{\text{func of } s}\,\underbrace{\sqrt{\int u(t)^2 dt}}_{\|u\|_2}$

so $\|Ku\|_2 = \sqrt{\int |(Ku)(s)|^2 ds} \le \|u\|_2 \sqrt{\int\int |k(s,t)|^2 dt\,ds}$ is upper bound for $\|K\|_2$. if $<\infty$, called Hilbert-Schmidt kernel

Numerical solution method : Nyström (1930)

2^{nd} kind
$$u(s) - \underbrace{\overbrace{\int_a^b K(s,t)\,u(t)\,dt}^{(Ku)(s)}}_{\text{quadr. } Q_n} = f(s) \qquad s \in [a,b]$$

Approx u by u_n which solves: $u_n(s) - \underbrace{\sum_{j=0}^n w_j\, k(s,t_j)\, u_n(t_j)}_{(K_n u_n)(s)} = f(s)$ (*)

where K_n is rank-n approx to op. K. its range is $\mathrm{Span}\{k(\cdot,t_j)\}$

Then values at nodes $u_i^{(n)} := u_n(s_i)$ sat. the lin. sys:
$$u_i^{(n)} - \sum_{j=0}^n w_j\, k(s_i,s_j)\, u_j^{(n)} = f(s_i) \qquad \text{(LS)}$$

ie $(I - A)\,u^{(n)} = f$ ← vector of RHS at nodes.

A: sq matrix; $ij^{\text{th}} = k(s_i,s_j)\,w_j$

So you've solved for u at nodes — how get full func $u_n(s)$?

Thm (12.11) If $\{u_i^{(n)}\}_{i=0}^n$ is soln. to (LS) then $u_n(s) = f(s) + \sum_{j=0}^n w_j\, k(s,t_j)\, u_j^{(n)}$ solves (*) (N)

pf: $u_n(s_j) = u_j^{(n)}$ $\forall j$ by construction of (LS) soln.
use to sub. for $u_j^{(n)}$ in (N) gives (*) QED. Subtle!

(*) expresses u_n as f + span {column slices of kernel at nodes $k(\cdot,t_j)$} , ie. interpolation basis.

(LS) is equiv. of Vandermonde sys to require interpolant agrees at nodes. , (N) reconstructs interpolant from node values.