Math 113
Homework Assignment Number Two
Due Wednesday, January 20^{th}

INSTRUCTIONS: As usual, for the "true/false" questions, just circle the correct answer. No justifications are required, but don't guess. You score is based on #right minus #wrong.

1. **TRUE or FALSE**: Every finite dimensional normed vector space is a Banach space.

2. **TRUE or FALSE**: Every compact Hausdorff space is a normal topological space.

3. **TRUE or FALSE**: If W is a closed subspace of a normed vector space V, then the quotient map $q : V \to V/W$ has norm one provided $W \neq V$.

4. **TRUE or FALSE**: Suppose that V and W are normed vector spaces and that $T : V \to W$ is linear. If V is finite dimensional, then T is bounded.

5. **TRUE or FALSE**: Let $C^1_{\mathbf{R}}([0,1])$ be the set of real-valued functions on $[0,1]$ with a continuous derivative on $[0,1]$. Let $\|f\| := \|f\|_\infty + \|f'\|_\infty$. Then $C^1_{\mathbf{R}}([0,1])$ is a Banach space with respect to $\|\cdot\|$.[1]

6. Work E.1.2.9 in the text.

7. Work E.2.1.1 in the text.

8. Define two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V to be equivalent in they determine the same topology on V. Prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if there are nonzero positive constants c and d such that

$$c\|v\|_1 \leq \|v\|_2 \leq d\|v\|_1 \quad \text{for all } v \in V.$$

9. (After Monday's lecture): Suppose that X is a locally compact Hausdorff space. Show that $C_0(X)$ is closed in $C^b(X)$ and that $C_c(X)$ is dense in $C_0(X)$.

[1] You may want to use the result that if $f_n \to f$ uniformly on $[0,1]$ and each f_n is differentiable with $f'_n \to g$ uniformly on $[0,1]$, then f is differentiable and $f' = g$. A proof of this statement follows easily from Theorem 7.17 of Rudin's *Principles of Real Analysis.*

10. Let X be a normed vector space and let $B = \{ x \in X : \|x\| \leq 1 \}$ be the *unit ball.* Show that if B is compact, then X is finite dimensional.[2] Since this is E.2.1.3 in the text, I was embarrassed not to be able to give a "quick" proof. You can either follow my steps below, or provide a better proof yourself.[3]

(a) Let $V = \{ x \in X : \|x\| < 1 \}$ be the open unit ball. Show that there is a finite set $\{ x_1, \dots, x_n \} \subset X$ such that

$$B \subset \bigcup_{i=1}^{n} x_i + \frac{1}{2}V.$$

(b) Let $Y = \operatorname{span}\{ x_1, \dots, x_n \}$ and conclude that

$$V \subset Y + \frac{1}{2}V$$

(c) Let

$$Z := \bigcap_{n} (Y + \frac{1}{2^n}V).$$

Observe that $V \subset Z$ and prove that $Z = Y$.

(d) Conclude that $Y = X$.

Remark. *I thought E.2.1.6, E.2.1.8, E.2.1.9 and E.2.1.10 all illustrated some interesting examples of Banach spaces, but I couldn't bear the thought of more to grade.*

[2] It is easy to go from here to showing that any normed vector space that is locally compact is necessarily finite dimensional.

[3] In fact, the "steps below" aren't my original ones. An student in the 2007 instance of this course, Chor Lam, came up with the "improved" version here. Can you find a better one?