WEAK TOPOLOGIES

MATH 113 - SPRING 2015

PROBLEM SET #5

Problem 1. Let E a Banach space, D a dense subset, $\{\varphi_n\}_{n\in\mathbb{N}}$ a sequence in E^* and $\varphi \in E^*$.

1. Prove that $$\varphi_n \xrightarrow[n\to\infty]{w^*} \varphi \Leftrightarrow \begin{cases} \{\varphi_n\}_{n\in\mathbb{N}} \text{ is bounded and} \\ \forall x \in D\ ,\ \langle \varphi_n, x\rangle \underset{n\to\infty}{\longrightarrow} \langle \varphi, x\rangle \end{cases}.$$

2. Can the boundedness assumption on $\{\varphi_n\}_{n\in\mathbb{N}}$ be removed?

Solution.

1. ($\Rightarrow$) The Uniform Boundedness principle implies that w^*-convergent sequences are bounded. ($\Leftarrow$) Since the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded, Alaoglu's Theorem implies the existence of a subsequence that converges to some $\psi \in E^*$, which, by the other assumption, coincides with φ on a dense subset.

2. No. Consider for instance $E = c_0(\mathbb{N})$, with dense subset D the subspace of finitely supported sequences and $\varphi_n = n \cdot \mathrm{eval}_n$. Then φ_n converges to 0 pointwise on D but not on E.

□

Problem 2. For $n \geq 1$ and $a \leq x \leq b$, let $f_n(x) = \sin(nx)$.

1. Prove that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to 0 in $L^2([a,b])$.

2. Does $\{f_n\}_{n\in\mathbb{N}}$ converge in $L^2([a,b])$?

Hint: 1. All bounded linear forms on $L^2([a,b])$ are of the form $f \mapsto \int_a^b f(x)g(x)\,dx$ with $g \in L^2([a,b])$ and step functions are dense in $L^2([a,b])$.

Solution. 1. This result goes by the name Riemann-Lebesgue Lemma. The equality $\lim_{n\to\infty}\int_a^b \sin(nx)f(x)\,dx = 0$ is obvious if f is a step function and the result follows by density.

2. If the sequence converged in $L^2([a,b])$ to some f, then it would converge weakly to the same limit, so f must be 0. However, the computation shows that $\|f_n\|_2^2 = \int_a^b \sin^2(nx)\,dx = \frac{b-a}{2} + \frac{\sin(2an)-\sin(2bn)}{4n} \underset{n\to\infty}{\not\to} 0.$

$\square$

Problem 3. Let $C_0(\mathbb{R}) = \{f \in C(\mathbb{R})\ ,\ \lim_{|x|\to\infty} f(x) = 0\}$.

1. Prove that $C_0(\mathbb{R})$ is closed in $L^\infty(\mathbb{R})$.

2. Describe how $L^1(\mathbb{R})$ can be seen as a subspace of $C_0(\mathbb{R})^*$.

3. Prove that every bounded sequence $\{u_n\}_{n\in\mathbb{N}}$ in $L^1(\mathbb{R})$ has a subsequence $\{u_{\varphi(n)}\}_{n\in\mathbb{N}}$ such that,

$$\forall f \in C_0(\mathbb{R})\ ,\ \lim_{n\to\infty}\int_{\mathbb{R}} u_{\varphi(n)}(x)f(x)\,dx \text{ exists.}$$

4. Find the w^*-limit in $C_0(\mathbb{R})^*$ of the sequence $\{n\chi_n\}_{n\geq 1}$, where χ_n is the indicator of the interval $\left[-\frac{1}{n},\frac{1}{n}\right]$.

Hint: 2. Remember the duality between L^p-spaces.

Solution. 1. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $C_0(\mathbb{R})$ that converges to some f in L^∞. The sequence being Cauchy in L^∞ and the f_n's being continuous, the sequence is also Cauchy in every $(C([a,b]), \|\cdot\|_\infty)$, from which it follows that f can be assumed continuous, up to changing of representative, and that the convergence is uniform. To prove that f vanishes at infinity, fix $\varepsilon > 0$ and chose n large enough to have $\|f_n - f\|_\infty < \frac{\varepsilon}{2}$. Since $f_n \in C_0(\mathbb{R})$, there is a compact outside of which $|f_n(x)| < \frac{\varepsilon}{2}$ so that $|f(x)|$ cannot exceed ε outside of that same compact.

2. The general idea is that taking dual spaces reverts inclusions:

$$\begin{array}{ccc} C_0(\mathbb{R}) & \subset & L^\infty(\mathbb{R}) \\ {}^*\downarrow & & \downarrow{}^* \\ C_0(\mathbb{R})^* & \supset & L^\infty(\mathbb{R})^* \underset{T}{\leftarrow} L^1(\mathbb{R}). \end{array}$$

Concretely, consider for $u \in L^1(\mathbb{R})$ the map $T_u(f) = \int_{\mathbb{R}} u(x)f(x)\,dx$ for $u \in C_0(\mathbb{R})$, then $|T_u(f)| \leq \|u\|_1 \|f\|_\infty$, so that $T_u \in C_0(\mathbb{R})^*$.

3. With T_u defined as before for $u \in L^1(\mathbb{R})$, observe that $\|T_u\| \leq \|u\|_1$. So the sequence T_{u_n} is bounded and Alaoglu's Theorem implies that it admits a w^*-convergent subsequence, which is exactly the expected result.

4. Let $f \in C_0(\mathbb{R})$. By continuity, for any $\varepsilon > 0$ there is an interval $[-\delta, \delta]$ on which $|f(x) - f(0)| \leq \varepsilon$. For $n > \delta^{-1}$, the average value of f on $\left[-\frac{1}{n}, \frac{1}{n}\right]$ (a.k.a $\frac{1}{2}T_{n\chi_n}(f)$) is within ε of $f(0)$. It follows that the w^*-limit of $\{T_{n\chi_n}\}_{n\in\mathbb{N}}$ is 2δ where δ denotes the Dirac measure at 0.

$\square$