WEAK TOPOLOGIES

MATH 113 - SPRING 2015

PROBLEM SET #5

Problem 1. Let E a Banach space, D a dense subset, $\{\varphi_n\}_{n\in\mathbb{N}}$ a sequence in E^* and $\varphi \in E^*$.

1. Prove that $$\varphi_n \underset{n\to\infty}{\overset{w^*}{\longrightarrow}} \varphi \Leftrightarrow \begin{cases} \{\varphi_n\}_{n\in\mathbb{N}} \text{ is bounded and} \\ \forall x \in D\ ,\ \langle \varphi_n, x\rangle \underset{n\to\infty}{\longrightarrow} \langle \varphi, x\rangle \end{cases}.$$
2. Can the boundedness assumption on $\{\varphi_n\}_{n\in\mathbb{N}}$ be removed?

Problem 2. For $n \geq 1$ and $a \leq x \leq b$, let $f_n(x) = \sin(nx)$.

1. Prove that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to 0 in $L^2([a,b])$.
2. Does $\{f_n\}_{n\in\mathbb{N}}$ converge in $L^2([a,b])$?

Hint: 1. All bounded linear forms on $L^2([a,b])$ are of the form $f \mapsto \int_a^b f(x)g(x)\,dx$ with $g \in L^2([a,b])$ and step functions are dense in $L^2([a,b])$.

Problem 3. Let $C_0(\mathbb{R}) = \{f \in C(\mathbb{R})\ ,\ \lim_{|x|\to\infty} f(x) = 0\}$.

1. Prove that $C_0(\mathbb{R})$ is closed in $L^\infty(\mathbb{R})$.
2. Describe how $L^1(\mathbb{R})$ can be seen as a subspace of $C_0(\mathbb{R})^*$.
3. Prove that every bounded sequence $\{u_n\}_{n\in\mathbb{N}}$ in $L^1(\mathbb{R})$ has a subsequence $\{u_{\varphi(n)}\}_{n\in\mathbb{N}}$ such that, $$\forall f \in C_0(\mathbb{R})\ ,\ \lim_{n\to\infty} \int_{\mathbb{R}} u_{\varphi(n)}(x) f(x)\,dx \text{ exists.}$$
4. Find the w^*-limit in $C_0(\mathbb{R})^*$ of the sequence $\{n\chi_n\}_{n\geq 1}$, where χ_n is the indicator of the interval $\left[-\frac{1}{n}, \frac{1}{n}\right]$.

Hint: 2. Remember the duality between L^p-spaces.