DUALITY

MATH 113 - SPRING 2015

PROBLEM SET #4

Problem 1. Let E and F be Banach spaces and $T : E \longrightarrow F$ a linear map such that

$$\forall \varphi \in F^* \quad , \quad \varphi \circ T \in E^*.$$

Prove that T is bounded.

Problem 2 (Closed convex sets that cannot be separated)**.** Let E_0 and F be the subsets of $\ell^1(\mathbb{N})$ defined by

$$E_0 = \left\{u \in \ell^1(\mathbb{N})\,,\, \forall n \geq 0\,,\, u_{2n} = 0\right\}$$

and

$$F = \left\{u \in \ell^1(\mathbb{N})\,,\, \forall n \geq 1\,,\, u_{2n} = 2^{-n} u_{2n-1}\right\}.$$

1. Verify that E_0 and F are closed subspaces and that $\overline{E_0 + F} = \ell^1(\mathbb{N})$.
2. Let v be the sequence defined by $v_{2n} = 2^{-n}$ and $v_{2n-1} = 0$.
   (a) Verify that v is in $\ell^1(\mathbb{N})$ and that $v \notin E_0 + F$.
   (b) Let $E = E_0 - v$. Prove that E and F are closed disjoint convex subsets of $\ell^1(\mathbb{N})$ that cannot be separated in the sense that there exists no couple $(\varphi, \alpha) \in (\ell^1(\mathbb{N}))^* \times \mathbb{R}$ such that $\varphi \neq 0$ and

   $$\varphi(e) \leq \alpha \leq \varphi(f)$$

   for all $e \in E$, $f \in F$.

Hints: 1. Finitely supported sequences are dense in $\ell^1(\mathbb{N})$. (See 4.(a) in Prob. 3.)
2.(b) What can be said of a functional that remains bounded on a linear subspace?

Problem 3 (Dual of $\ell^p(\mathbb{N})$)**.** In this problem, we assume the sequences to be real-valued. Let $p \in [1, +\infty)$ and denote by q the only element of $(1, +\infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The purpose of this problem is to identify $\ell^p(\mathbb{N})^*$ with $\ell^q(\mathbb{N})$.
To this end, we shall prove that the map Φ defined on ℓ^q by

$$\Phi(u)v = \sum_{n \geq 0} u_n v_n$$

is an isometry.

1. Verify that $\Phi(u)$ is a linear functional on $\ell^p(\mathbb{N})$ for each $u \in \ell^q(\mathbb{N})$ and that Φ is linear.

2. Let $u \in \ell^q(\mathbb{N})$. Prove that $\Phi(u) \in \ell^p(\mathbb{N})^*$ and that $\|\Phi(u)\| \leq \|u\|_q$.

3. Let $u \in \ell^q(\mathbb{N})$ be fixed.

   (a) Assume $p > 1$. Verify that the sequence v defined by

   $$v_n = \|u\|_q^{1-q} \operatorname{sign}(u_n) \, |u_n|^{q-1}$$

   is in ℓ^p and compute $\Phi(u)v$.

   (b) Let $p = 1$. For $\varepsilon > 0$, find v on the unit sphere of $\ell^1(\mathbb{N})$ such that $|\Phi(u)v| > \|u\|_\infty - \varepsilon$.

   (c) What have we proved so far?

4. (a) Prove that finitely supported sequences are dense in $\ell^p(\mathbb{N})$ for $p \geq 1$.

   (b) Does the result hold in $\ell^\infty(\mathbb{N})$?

5. For $n \in \mathbb{N}$, define the sequence e^n by $e^n_k = \delta_{k,n}$, that is $e^n = \{\overbrace{0, 0, \ldots, 0}^{n}, 1, 0, 0, \ldots\}$. Let $\varphi \in \ell^p(\mathbb{N})^*$ and $\gamma_n = \varphi(e^n)$. For $N \in \mathbb{N}$, define a sequence δ^N by

   $$\delta^N = \{\gamma_0|\gamma_0|^{q-2}, \gamma_1|\gamma_1|^{q-2}, \ldots, \gamma_N|\gamma_N|^{q-2}, 0, 0, \ldots\}.$$

   (a) Compute $\varphi(\delta^N)$.

(b) Prove that $\sum_{n=0}^{N} |\gamma_n|^q \leq \|\varphi\| \left(\sum_{n=0}^{N} |\gamma_n|^q\right)^{\frac{1}{p}}$

(c) Deduce that the N-truncation of the sequence $\gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ has norm less than $\|\varphi\|$ in $\ell^q(\mathbb{N})$.

(d) Conclude that γ is in $\ell^q(\mathbb{N})$.

6. Verify that $\varphi(u) = \Phi(\gamma)(u)$ if u is finitely supported and conclude.

7. Prove the existence of a bounded linear functional on $\ell^\infty(\mathbb{N})$ that is not of the form $\Phi(u)$ with $u \in \ell^1(\mathbb{N})$.
Hint: consider the subspace C of convergent sequences and study the map $\lambda : v \mapsto \lim_{n\to\infty} v_n$.