LINEAR OPERATORS ON BANACH SPACES

MATH 113 - SPRING 2015

PROBLEM SET #3

Problem 1. Let E be the space $C([0,1])$ equipped with $\|\cdot\|_\infty$. If f is differentiable, we write $D(f) = f'$.

1. Let F be a closed subspace of E that is included in $C^1([0,1])$.
   (a) Show that $D : F \longrightarrow E$ is Lipschitz.
   (b) Prove that F is finite dimensional.
2. Let $G = (C^1([0,1]), \|\cdot\|_\infty)$.
   (a) Show that $D : G \longrightarrow E$ is closed.
   (b) Is it continuous?

Hints: 1.(a) Study the graph of D. - 1.(b) Study the unit ball of F.

Solution. 1. (a) Both E and F are Banach spaces and D is linear so by the Closed Graph Theorem, it suffices to prove that D is closed. Let $\{f_n\}$ be a sequence in F such that f_n and $Df_n = f_n'$ converge, say to f and g respectively. This means that $f_n \to f$ and $f_n' \to g$ uniformly, which implies that $f' = g$, so that (f, g) belongs to the graph of D, which is therefore continuous, hence Lipschitz.

(b) By Riesz's Theorem, it is enough to prove that the closed unit ball in F is compact. It is bounded by definition and closed in $E = C([0,1])$ so, it suffices to prove that it is equicontinuous. Let C be a Lipschitz constant for D. Then $\|f'\|_\infty \leq C\|f\|_\infty \leq C$ for all f in the unit ball of F, which is therefore uniformly equicontinuous.

2. (a) This was already done in 1.(a): take $F = G$.

   (b) No: consider the sequence $\{f_n : x \longmapsto x^n\}$ in the unit ball of G and its image under D. Note that this does not contradict the Closed Graph Theorem, since G is not closed in E and therefore not complete.

□

Problem 2. Let E be a normed linear space, F a closed subspace of E and

$$\pi : E \longrightarrow E/F$$

the natural surjection.

1. Let $x \in E$ and $r > 0$. Show that $\pi(B(x, r)) = B(\pi(x), r)$.
2. Let $\mathcal{U}$ be a subset of E/F. Prove that $\mathcal{U}$ is open if and only if $\pi^{-1}(\mathcal{U})$ is open in E.
3. Prove that π is an open map.
4. Show that the Open Mapping Theorem can be deduced from the Bounded Inverse Theorem.

Solution. 1. First, observe that translations are isometries that commute to π so we may assume $x = 0$. The surjection π is 1-Lipschitz by definition of the norm on E/F so $\pi(B(0, r)) \subset B(\pi(0), r)$.
Conversely, assume that $y \in E/F$ has norm $< r$. Choose $x \in E$ such that $\pi(x) = y$. Then $\|y\| = \inf_{v \in F} \|x + v\| < r$ so there exists $v \in F$ such that $\|x + v\| < r$ and y has a preimage in $B(0, r)$, which means that $B(\pi(0), r) \subset \pi(B(0, r))$.

2. Again, π being 1-Lipschitz, it is continuous, which implies that if $\mathcal{U}$ is open in E/F, then $\pi^{-1}(\mathcal{U})$ is open in E.
For the converse, assume that $\pi^{-1}(\mathcal{U})$ is open in E and let $y \in \mathcal{U}$, with preimage $x \in E$. Since $\pi^{-1}(\mathcal{U})$ is open and contains x, there exists $r > 0$ such that $B(x, r) \subset \pi^{-1}(\mathcal{U})$.
By the result of the previous question, $\mathcal{U} = \pi(\pi^{-1}(\mathcal{U}))$ contains the ball $B(y, r)$ so it is a neighborhood of y.

3. Let U be open in F. By the result of the previous question, in order to prove that $\pi(U)$ is open in E/F, it suffices to prove that $\pi^{-1}(\pi(U))$ is open in E, which follows from the observation that $\pi^{-1}(\pi(U)) = U+F = \bigcup_{v\in F} U+v$.

4. Let $T : E \longrightarrow F$ be a surjective continuous linear map between Banach spaces. Consider the induced map $\tilde{T} : E/\ker T \longrightarrow F$. Apply the Bounded Inverse Theorem to $\tilde{T}$ and conclude by noticing that $T = \tilde{T} \circ \pi$.

$\square$

Problem 3 (Bilinear maps)**.** Let E_1, E_2 and F be normed linear spaces and equip $E_1 \times E_2$ with the norm $\|(x,y)\| = \max(\|x\|, \|y\|)$. A map $B : E_1 \times E_2 \longrightarrow F$ is said **bilinear** if all the maps

$$\begin{array}{rcl} \Lambda_x : E_2 & \longrightarrow & F \\ y & \longmapsto & B(x,y) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathrm{P}_y : E_1 & \longrightarrow & F \\ x & \longmapsto & B(x,y) \end{array}$$

are linear. Moreover, B is said

- **separately continuous** if all the maps Λ_x and P_y are continuous;
- **bounded** if

$$\|B\| := \sup\{\|B(x,y)\| \quad , \quad x \in E_1\ ,\ y \in E_2\ ,\ \|x\| \leq 1\ ,\ \|y\| \leq 1\} < \infty.$$

1. Show that the statements
   (a) B is bounded.
   (b) There exists a constant $C \geq 0$ such that $\|B(x,y)\| \leq C\|x\|\|y\|$ for all (x,y) in $E_1 \times E_2$.
   (c) B is continuous.
   (d) B is continuous at $(0,0)$.

   are equivalent and that if they hold, $\|B\|$ is the smallest C satisfying (b).

Recall that the set of bounded linear maps between linear spaces E and F is denoted by $\mathcal{L}(E,F)$. The set of bounded bilinear maps from $E_1 \times E_2$ to F will be denoted by $\mathcal{B}(E_1 \times E_2, F)$.

2. Let E and F be normed linear spaces. Show that the map

$$\begin{array}{rcl} \beta : \mathcal{L}(E,F) \times E & \longrightarrow & F \\ (T,x) & \longmapsto & T(x) \end{array}$$

is in $\mathcal{B}\left(\mathcal{L}(E,F) \times E, F\right)$ and that $\|\beta\| \leq 1$.

3. Let E, F and G be normed linear spaces. Show that the map

$$\begin{array}{rcl} \gamma : \mathcal{L}(F,G) \times \mathcal{L}(E,F) & \longrightarrow & \mathcal{L}(E,G) \\ (S,T) & \longmapsto & S \circ T \end{array}$$

is in $\mathcal{B}\left(\mathcal{L}(F,G) \times \mathcal{L}(E,F), \mathcal{L}(E,G)\right)$ and that $\|\gamma\| \leq 1$.

4. Show that $\mathcal{B}(E_1 \times E_2, F)$ equipped with the pointwise operations and $\|\cdot\|$ defined above is a normed linear space.

5. (a) Show that $\mathcal{B}(E_1 \times E_2, F)$ is isometrically isomorphic to $\mathcal{L}\left(E_1, \mathcal{L}(E_2, F)\right)$.
   (b) What can be said of $\mathcal{B}(E_1 \times E_2, F)$ if F is a Banach space?

6. Assume that E_1 and E_2 are Banach spaces. Show that a bilinear map $B : E_1 \times E_2 \longrightarrow F$ is bounded if and only if it is separately continuous.

7. Consider $E = \mathbb{R}[X]$ equipped with the norm $\|P\| = \displaystyle\int_0^1 |\tilde{P}(x)|\,dx$ where $\tilde{P}$ is the function associated with the polynomial P. Show that the bilinear map α defined on $E \times E$ by $\alpha(P,Q) = \displaystyle\int_0^1 \tilde{P}(x)\tilde{Q}(x)\,dx$ is separately continuous but not bounded.

Solution. The equivalences in 1. and the statement in 4. can be proved in the same fashion as the analogous ones in the case of linear maps. The results in 2. and 3. are direct consequences of the properties of the operator norm.

5. (a) Consider the map $x \longmapsto \Lambda_x$.
   (b) It is a Banach space.

6. The implication (bounded $\Rightarrow$ separately continuous) is trivial. Conversely, assume B separately continuous and fix x in E_1 with $\|x\| \leq 1$. Then $|\Lambda_x(y)| = |B(x,y)| \leq \|\mathrm{P}_y\|$ for all y in E_2. By the Uniform Boundedness Principle, the family $\{\Lambda_x\ ,\ \|x\| \leq 1\}$ is bounded in $\mathcal{L}(E_2, \mathbb{R})$ so there

exists a constant C such that $|B(x,y)| = |\Lambda_x(y)| \leq C$ for all x, y in the closed unit ball of $E_1 \times E_2$. Note that it suffices to assume only one of the E_i to be complete for the argument to work.

7. Separate continuity follows from the fact that $\|\Lambda_P\| \leq \|P\|_\infty$ and the symmetry of α. For $n \geq 1$, the polynomial nX^n lies on the unit sphere of E and $\alpha(nX^n, nX^n) = \dfrac{n^2}{2n+1} \underset{n\to\infty}{\longrightarrow} +\infty$ so α is not bounded.

$\square$