Applications of the Arzelà-Ascoli and the Baire Category Theorems

Math 113 - Spring 2015

Problem set #2

Problem 1 (Hölder maps)**.**
A function $f \in C([0,1],\mathbb{R})$ is said to be α-*Hölder* if

$$h_\alpha(f) = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

is finite. For $M > 0$ and $0 < \alpha \leq 1$, denote

$$H_{\alpha,M} = \{f \in C([0,1],\mathbb{R})\ ,\ h_\alpha(f) \leq M \text{ and } \|f\|_\infty \leq M\}\,.$$

Prove that $H_{\alpha,M}$ is compact in $(C([0,1],\mathbb{R}), \|\cdot\|_\infty)$.

Problem 2. Show that a normed linear space over $\mathbb{R}$ that has a countable algebraic basis cannot be complete.

Problem 3. Let $f : [0,+\infty) \longrightarrow \mathbb{R}$ be continuous and assume that for all $x > 0$,

$$\lim_{n\to\infty} f(nx) = 0.$$

Prove that $\lim_{x\to\infty} f(x) = 0$.

Hint: for $\varepsilon > 0$ and $n \in \mathbb{N}$, consider $F_{n,\varepsilon} = \{x \geq 0\ ,\ \forall p \geq n\ ,\ |f(px)| \leq \varepsilon\}$.

Problem 4. Show that nowhere differentiable functions are dense in $E = C([0,1],\mathbb{R})$ equipped with its ordinary norm.

Hint: consider, for $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$U_{n,\varepsilon} = \left\{ f \in E\ ,\ \forall x \in [0,1]\ ,\ \exists y \in [0,1]\ ,\ |x-y| < \varepsilon \quad \text{and} \quad \left|\frac{f(y)-f(x)}{y-x}\right| > n \right\}.$$