METRIC SPACES

MATH 113 - SPRING 2015

PROBLEM SET #1

Problem 1 (Distance to a subset and metric Urysohn's Lemma)**.**
Let (E, d) be a metric space. For any subset $A \subset E$ and any point $x \in E$, the *distance* between x and A is defined by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

1. Show that $d(x, A) = d(x, \bar{A})$.

2. Show that $d(\cdot, A)$ is 1-Lipschitz.

3. Let A and B be disjoint closed subsets of E. Prove the existence of a continuous function $f : E \longrightarrow \mathbb{R}$ such that:

   (a) $0 \leq f(x) \leq 1$ for all $x \in E$;

   (b) $f(x) = 0$ for all $x \in A$;

   (c) $f(x) = 1$ for all $x \in B$.

Solution. 1. Observe that $A \subset \bar{A}$ so $d(x, A) \geq d(x, \bar{A})$. For the other inequality, consider α in $\bar{A}$. There exists a sequence $\{a_n\} \in A^{\mathbb{N}}$ that converges to α. Given x fixed, the function $d(x, \cdot)$ is continuous so $\lim_{n\to\infty} d(x, a_n) = d(x, \alpha)$. Since $d(x, a_n) \geq d(x, A)$ for every n, it follows that $d(x, \alpha) \geq d(x, A)$. This is true for every α in $\bar{A}$ so $d(x, \bar{A}) \geq d(x, A)$.

2. For $x, y \in E$ and $a \in A$, the triangle inequality and the definition of $d(x, A)$ imply that $d(x, A) \leq d(x, y) + d(y, a)$. This is true for every $a \in A$ so $d(x, A) \leq d(x, y) + d(y, A)$ and we get $d(x, A) - d(y, A) \leq d(x, y)$. The same argument gives $d(y, A) - d(x, A) \leq d(x, y)$ hence the result.

3. Consider $x \longmapsto \dfrac{d(x,A)}{d(x,A)+d(x,B)}$.

$\square$

Problem 2 (Completeness is not a topological property)**.**
Let $E = (0,+\infty)$ and for $x, y \in E$, consider $\delta(x,y) = \left|\dfrac{1}{x} - \dfrac{1}{y}\right|$.

1. Prove that δ is a distance on E and that it induces the same topology as the Euclidean distance d.

2. Is the map $x \mapsto x^{-1}$ uniformly continuous as a map from (E,d) to itself ? As a map from (E,d) to (E,δ)?

3. Is (E,δ) complete ? What about $((0,1],d)$ and $((0,1],\delta)$?

Solution. 1. *Method 1*: prove that every open d-ball contains a δ-ball with the same center and vice versa. *Method 2*: prove that $(E,d) \xrightarrow{\text{Id}} (E,\delta)$ is a homeomorphism. To see this, it is convenient to decompose the identity map as $(E,d) \xrightarrow{\varphi} (E,d) \xrightarrow{\varphi} (E,\delta)$ where $\varphi(x) = x^{-1}$ and prove that both are homeomorphisms. Note that both methods boil down to the fact that φ is a homeomorphism from $(E, d \,\text{or}\, \delta)$ to $(E, d \,\text{or}\, \delta)$.

2. No. Yes.

3. No: $u_n = n$ is Cauchy but it does not converge (argue by contradiction). No: it is not closed in $(\mathbb{R},d)$. Alternatively, consider $u_n = \frac{1}{n}$, Cauchy but not convergent in $(0,1]$.
Yes. *Method 1*: show that a Cauchy sequence $\{u_n\}$ for δ is also Cauchy for d hence converges for d in the closure of $(0,1]$. If the d-limit is > 0, it is also the δ-limit because d and δ induce the same topology (or check it directly with balls) so the sequence converges. Assume the limit is 0. Then $\delta(1,u_n)$ diverges to $+\infty$ so $\{u_n\}$ is not bounded which is impossible since it is Cauchy. *Method 2*: $x \mapsto x^{-1}$ is an isometry (hence uniformly continuous) between $((0,1],\delta)$ and $([1,+\infty),d)$, which is closed in $(\mathbb{R},d)$ complete, so is complete.

$\square$

Problem 3 (The Banach Contraction Principle)**.**
Let (E, d) be a complete metric space and $f : E \longrightarrow E$.

1. Show that if f is k-Lipschitz with $k < 1$, the equation $f(x) = x$ has a unique solution in E.

2. Show that if E is compact, it is enough to have $d(f(x), f(y)) < d(x, y)$ for all x, y to obtain the same result.

Solution. 1. Think triangle inequality and geometric series.

2. The real-valued function $x \mapsto d(x, f(x))$ is continuous on a compact set so it is bounded and the extrema are attained.

□

Problem 4 (Completeness of $\ell^2(\mathbb{N})$)**.**
Show that the set of sequences $U = \{u_n\}$ such that $\sum_{n \geq 0} |u_n|^2$ converges is complete for the norm $\|U\|_2 = \left(\sum_{n=0}^{\infty} |u_n|^2 \right)^{\frac{1}{2}}$

Solution. The skeleton of the proof we studied for the space of bounded functions with values in a complete space carries over. □

Problem 5 (Cantor's Intersection Theorem)**.**
Let (E, d) be a metric space and $A \subset E$ a non-empty subset. The *diameter* of A is defined by

$$\operatorname{diam}(A) = \sup_{x,y \in A} d(x, y).$$

Prove that E is complete if and only if for every decreasing sequence $\{F_n\}_{n \in \mathbb{N}}$ of closed subsets of E such that $\lim_{n \to \infty} \operatorname{diam}(F_n) = 0$, there is a point x such that

$$\bigcap_{n \in \mathbb{N}} F_n = \{x\}.$$

Solution. See Section 9.4 of [Royden-Fitzpatrick].

□

Problem 6 (Characterizations of compactness for metric spaces)**.**
Let (E, d) be a metric space. Prove that the following conditions are equivalent.

(i) E has the *Borel-Lebesgue* property, *i.e.* is topologically compact.

(ii) If $\mathcal{F}$ is a family of closed subsets of E such that every subfamily has nonempty intersection, then $\bigcap_{F \in \mathcal{F}} F \neq \varnothing$.

(iii) E is complete and *totally bounded i.e.* can be covered by finitely many open balls of radius ε, for any $\varepsilon > 0$.

(iv) E has the *Bolzano-Weierstrass* property, *i.e.* is sequentially compact.

Solution. The equivalence between (i) and (ii) holds in topological (non-necessarily metric) spaces. See Propositions 17, 18 and 19 in Section 9.5 of [Royden-Fitzpatrick] for the rest. □