MATH 113 - ANALYSIS
SPRING 2015
TAKE-HOME MIDTERM

ELEMENTS OF SOLUTION

The goal of this problem is to give a proof of the following density result.

Theorem (Weierstrass). *Every continuous function on a segment of the real line is the uniform limit of a sequence of polynomial functions.*

0. The theorem asserts in particular that the family of functions $\{x \mapsto x^n\}_{n\in\mathbb{N}}$ is a topological basis of $(C([0,1]), \|\cdot\|_\infty)$. Is it an algebraic basis?

No: linear combinations of monomials are smooth while some continuous functions fail to be differentiable.

Let $\mathcal{E}$ be the space of continuous and compactly supported complex-valued functions on $\mathbb{R}$. For $f, g \in \mathcal{E}$, let $f \star g$ denote the *convolution product of* f *and* g, defined by

$$f \star g\,(x) = \int_{\mathbb{R}} f(t)g(x-t)\,dt.$$

1. Verify that $(\mathcal{E}, +, \star)$ is an algebra. Is it unital?

The verification is routine, using Fubini and changes of variables. Note that $\mathrm{supp}(f * g) \subset \mathrm{supp}(f) + \mathrm{supp}(g)$. Assume that $(\mathcal{E}, +, \star)$ is unital. Then, there exists a continuous function f such that $f * g = g$ for all $g \in \mathcal{E}$. In particular, the relation $f * g(0) = g(0)$ implies that $\int_{\mathbb{R}} f(t)h(t)\,dt = h(0)$ for any $h \in \mathcal{E}$. Since f cannot be identically zero, assume that it takes a positive value at $x_0 \neq 0$. Then there exists $\delta > 0$ such that $0 < x_0 - \delta$ and f only takes positive values on $I = [x_0 - \delta, x_0 + \delta]$. Consider h supported in I, non-negative and not identically zero. Then $h(0) = 0 \neq \int_{\mathbb{R}} f(t)h(t)\,dt$, which contradicts the assumption on f. Therefore f must vanish everywhere except perhaps at 0, but since it must be continuous, it is constantly zero.

Definition. An *approximate unit* in $\mathcal{E}$ is a sequence $\{\chi_n\}_{n\geq 1}$ such that for any f in $\mathcal{E}$, the sequence $\{\chi_n \star f\}$ converges uniformly to f.

2. Prove that a sequence of non-negative functions α_n in $\mathcal{E}$ such that

$$\forall n \geq 1 \quad , \quad \int_{\mathbb{R}} \alpha_n(t)\,dt = 1 \qquad \textbf{and} \qquad \forall A > 0 \quad , \quad \lim_{n\to\infty} \int_{|t|\geq A} \alpha_n(t)\,dt = 0$$

is an approximate unit.

Let $f \in \mathcal{E}$. Since f is continuous and compactly supported, it is uniformly continuous. Fix $\varepsilon > 0$ and let $\eta > 0$ be such that

$$|x-y| < \eta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Then, if n is large enough so that $\displaystyle\int_{|t|\geq\eta} \alpha_n(t)\,dt < \varepsilon$,

$$\begin{aligned}
|f * \alpha_n(x) - f(x)| &= \left|\int_{\mathbb{R}} (f(x-t) - f(x))\,\alpha_n(t)\right| dt \\
&\leq \int_{|t|>\eta} |f(x-t) - f(x)|\alpha_n(t)\,dt + \int_{-\eta}^{\eta} |f(x-t) - f(x)|\alpha_n(t)\,dt \\
&< 2\|f\|_\infty \varepsilon + \varepsilon \int_R \alpha_n(t)\,dt = (2\|f\|_\infty + 1)\varepsilon,
\end{aligned}$$

which can be made arbitrarily small, independently of x.

3. Define, for $n \geq 1$, $a_n = \displaystyle\int_{-1}^{1} (1-t^2)^n\,dt$ and $p_n : t \longmapsto \begin{cases} \dfrac{(1-t^2)^n}{a_n} & \textbf{if } |t| \leq 1 \\ 0 & \textbf{otherwise} \end{cases}$.

Show that $\{p_n\}_{n\geq 1}$ is an approximate unit in $\mathcal{E}$.

The non-negativity and normalization are immediate. Note that $\displaystyle\int_{|t|\geq A} p_n(t)\,dt = 0$ if $A \geq 1$ and that

$$a_n = 2\int_0^1 (1-t^2)^n\,dt \geq 2\int_0^1 (1-t)^n\,dt = \frac{2}{n+1}.$$

For $0 < A < 1$ and $n \geq 1$, we see that

$$\begin{aligned}
\int_{|t|\geq A} p_n(t)\,dt &= \frac{2}{a_n}\int_A^1 (1-t^2)^n\,dt \\
&\leq \frac{2}{a_n}(1-A^2)^n \\
&= (n+1)\underbrace{(1-A^2)}_{<1}{}^n \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

so $\{p_n\}_{n\geq 1}$ is an approximate unit in $\mathcal{E}$.

4. Let f be a function in $\mathcal{E}$ that vanishes outside of $[-\frac{1}{2}, \frac{1}{2}]$. Prove that, for every $n \geq 1$, the function $p_n \star f$ is polynomial on its support.

First observe that $p_n(x-t)$ is a polynomial in x. To fix notations, we write

$$p_n(x-t) = \sum_{k=0}^{2n} c_k(t) x^k.$$

Then, for x in the support of the convolution,

$$(f * p_n)(x) = \sum_{k=0}^{2n} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) c_k(t)\, dt \right) x^k$$

which is a polynomial expression.

5. Prove Weierstrass' Theorem.

It follows from the previous results that a continuous function with compact support in $[-\frac{1}{2}, \frac{1}{2}]$ is a uniform limit of polynomial functions. Now let f be a continuous function defined on a segment $[a, b]$. Extend f to a function $\tilde{f} \in \mathcal{E}$. This can be done for instance by requesting that $\tilde{f}$ be 0 oustide of $[a-1, b+1]$, coincide with f on $[a, b]$ and affine elsewhere.

An affine transformation from $[a-1, b+1]$ to $[-\frac{1}{2}, \frac{1}{2}]$ allows to use the result proved in **4.** and to conclude.