MATH 113 - ANALYSIS
SPRING 2015
TAKE-HOME MIDTERM

DUE MAY 1^{ST}

The goal of this problem is to give a proof of the following density result.

Theorem (Weierstrass)**.** *Every continuous function on a segment of the real line is the uniform limit of a sequence of polynomial functions.*

0. The theorem asserts in particular that the family of functions $\{x \mapsto x^n\}_{n\in\mathbb{N}}$ is a topological basis of $(C([0,1]), \|\cdot\|_\infty)$. Is it an algebraic basis?

Let $\mathcal{E}$ be the space of continuous and compactly supported complex-valued functions on $\mathbb{R}$. For $f, g \in \mathcal{E}$, let $f \star g$ denote the *convolution product of f and g*, defined by

$$f \star g\,(x) = \int_{\mathbb{R}} f(t)g(x-t)\,dt.$$

1. Verify that $(\mathcal{E}, +, \star)$ is an algebra. Is it unital?

Definition. An *approximate unit* in $\mathcal{E}$ is a sequence $\{\chi_n\}_{n\geq 1}$ such that for any f in $\mathcal{E}$, the sequence $\{\chi_n \star f\}$ converges uniformly to f.

2. Sketch the graphs of functions α_n in $\mathcal{E}$ such that

- $\forall n \geq 1$, α_n only takes non-negative values;
- $\forall n \geq 1$, $\displaystyle\int_{\mathbb{R}} \alpha_n(t)\,dt = 1$;
- $\forall A > 0$, $\displaystyle\lim_{n\to\infty} \int_{|t|\geq A} \alpha_n(t)\,dt = 0$;

and prove that the sequence $\{\alpha_n\}_{n\geq 1}$ is an approximate unit in $\mathcal{E}$.

3. Define, for $n \geq 1$, $a_n = \displaystyle\int_{-1}^{1} (1-t^2)^n\,dt$ and $p_n : t \longmapsto \begin{cases} \dfrac{(1-t^2)^n}{a_n} & \text{if } |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

Show that $\{p_n\}_{n\geq 1}$ is an approximate unit in $\mathcal{E}$.

4. Let f be a function in $\mathcal{E}$ that vanishes outside of $[-\frac{1}{2}, \frac{1}{2}]$. Prove that, for every $n \geq 1$, the function $p_n \star f$ is polynomial on its support.

5. Prove Weierstrass' Theorem.