MATH 113 - ANALYSIS
SPRING 2015
FINAL EXAMINATION

DURATION: 4 HOURS

This exam consists of five independent problems. You may treat them in the order of your choosing.

If you were not able to solve a question but wish to use the result to solve another one, you are welcome to do so, as long as you indicate it explicitly.

Problem 1

Let $\{\alpha_n\}_{n\geq 0}$ be a bounded sequence of complex numbers and S the map defined on $\ell^2(\mathbb{N})$ by

$$S(u_0, u_1, \ldots) = (0, \alpha_0 u_0, \alpha_1 u_1, \ldots).$$

1. Verify that $S \in \mathcal{B}(\ell^2(\mathbb{N}))$ and compute its operator norm.

2. Determine the adjoint of S.

3. Is S a normal operator?

4. Are there sequences $\{\alpha_n\}_{n\geq 0}$ such that S is an isometric embedding? An isometry?

Problem 2

Let $\mathcal{H}$ be a separable Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. Define an operator $A^\dagger$ on $\mathcal{H}^*$ by

$$A^\dagger \varphi = \varphi \circ A.$$

1. Verify that $A^\dagger \in \mathcal{B}(\mathcal{H}^*)$.

2. What is the relation between $A^\dagger$ and the adjoint A^* of A?

Problem 3

Let $\mathcal{H}$ be a separable Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Recall that if T is **hermitian**, then

$$\|T\| = \sup\{|\langle T\xi \,|\, \xi\rangle|\,,\ \|\xi\| = 1\}.$$

1. Assume T hermitian such that $T\xi \perp \xi$ for all $\xi \in \mathcal{H}$.

a. What can be said of T?

b. Does the result hold if T is not hermitian?

Let A be an arbitrary element of $\mathcal{B}(\mathcal{H})$

2. Find an operator $B \in \mathcal{B}(\mathcal{H})$ such that

$$\|A\xi\|^2 - \|A^*\xi\|^2 = \langle B\xi \,|\, \xi\rangle$$

for all $\xi \in \mathcal{H}$.

3. Prove that A is normal if and only if $\|A\xi\| = \|A^*\xi\|$ for all $\xi \in \mathcal{H}$.

Problem 4

Let $\mathcal{A}$ be a unital Banach algebra, with unit denoted by 1, and a, b elements of $\mathcal{A}$.

1. Let $\lambda \in \mathbb{C}^\times$ be such that $\lambda - ab$ is invertible. Prove that $\lambda - ba$ is invertible, with inverse

$$\lambda^{-1} + \lambda^{-1} b(\lambda - ab)^{-1} a.$$

2. Prove that $\mathrm{Sp}_{\mathcal{A}}(ab) \cup \{0\} = \mathrm{Sp}_{\mathcal{A}}(ba) \cup \{0\}$.

3. Prove that ab and ba have the same spectral radius.

4. Give an example of elements a and b in a Banach algebra such that $\mathrm{Sp}_{\mathcal{A}}(ab) \neq \mathrm{Sp}_{\mathcal{A}}(ba)$.

Problem 5

Let E and F be Banach spaces.

1. Prove that every weakly convergent sequence in E is bounded.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a weakly convergent sequence in E, with weak limit x and $T : E \longrightarrow F$ a bounded linear map.

2. Prove that the sequence $\{Tx_n\}_{n\in\mathbb{N}}$ converges weakly to Tx.

From now on, assume that T is **compact**, that is, the image by T of the closed unit ball of E is compact in F.

3. Show that every subsequence of $\{Tx_n\}_{n\in\mathbb{N}}$ has a subsequence that converges (strongly) to Tx.

4. Conclude that Tx_n converges strongly to Tx.