Math 113: Functional Analysis

Syllabus, problems and solutions

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References

The main sources for the material presented in class were:

- [RF] *Real Analysis* (Fourth edition), by H. Royden and P. Fitzpatrick.
- [C] *A Course in Abstract Analysis*, by J. Conway.
- [SS] *Fourier Analysis: an introduction*, E. M. Stein and R. Shakarchi

Effective syllabus

I. Metric spaces - *6 Lectures and 2 problem sessions*

I.1. Definitions and examples
I.2. Metric topology
I.3. Complete spaces
I.4. The Ascoli-Arzelà Theorem
I.5. The Baire Category Theorem

II. Banach spaces - *15 Lectures and 4 problem sessions*

II.1. Normed linear spaces
II.2. Banach spaces
II.3. Linear operators between Banach spaces
II.4. Hahn-Banach Theorems, duality
II.5. Banach algebras, abstract C^*-algebras

III. Hilbert spaces - *3 Lectures and 1 problem session*

III.1. Definitions
III.2. The Riesz Representation Theorem
III.3. Hilbert bases
III.4. Adjoints, concrete C^*-algebras

IV. Elements of Harmonic Analysis - *3 Lectures and 1 problem session*

IV.1. Fourier series
IV.2. Elements of Representation Theory
IV.3. Fourier Analysis on groups

Contents

Week 1

Lecture 1. Metrics, norms on a linear space.
Examples: $\mathbb{R}^n$ Euclidean, discrete metric, L^p norms (on $\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$ and (X,μ)).
Semi-norms, pseudo-metrics, separation.
Examples: $\|\cdot\|_\infty$ on $C(X)$, $C_0(X)$ and $C_b(X)$.
Metric subspaces and metric products. Equivalent metrics.
Fun: Manhattan distance and distance SNCF, ultrametric distances, p-adic norm on $\mathbb{Q}$.

Lecture 2. Isometric spaces.
General topology reminder: topological spaces, neighborhoods, interior, closure, continuous maps, homeomorphisms, compact spaces.
Metric topology: open and closed balls, description of neighborhoods, interior and closure.
$\overline{\mathcal{B}(x,r)} \neq \mathcal{B}_c(x,r)$ in general (discrete metric). Equality holds in normed linear spaces.
Sequences in metric spaces: convergence, sequential characterization of closure, of continuity, compact metric spaces.
Equivalent metrics define the same topology. The converse does not hold: d and $\min(d,1)$ are topologically equivalent but not equivalent in general.
Fun: quasi-isometries. ($\mathbb{Z}^n \overset{QI}{\sim} \mathbb{R}^n$), $(\mathbb{R}^2, \text{Eucl.}) \overset{QI}{\sim} (\mathbb{R}^2, \text{Manhatt.})$.

Lecture 3. Cauchy sequences and complete spaces.
For any set X, $B(X,\mathbb{R})$ is complete for $\|\cdot\|_\infty$. Generalization to continuous functions with values in a complete normed linear space.
Non-example: $C([0,1],\mathbb{R})$ is not complete for $\|\cdot\|_1$.
L^p spaces are complete. A subset of a complete metric space is complete iff it is closed.
Uniform continuity and Lipschitz maps. Example: $f \longmapsto \int_a^b f$.
Fun: in a complete ultrametric space, $\sum u_n$ converges $\Leftrightarrow \lim u_n = 0$.

Problem session 1. Distance to a subset and Urysohn's Lemma.
Study of the distance $\delta(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ on $(0,+\infty)$ and $(0,1]$. Topology, completeness.
Solutions presented by: Sam.

Week 2

Lecture 4. Density, separability.
Compact metric spaces are separable.
Completion of a metric space: construction, uniqueness up to a unique isometry.

Lecture 5. Reminder on compact metric spaces: compact $\Leftrightarrow$ complete and totally bounded $\Leftrightarrow$ sequentially compact, Extreme Value Theorem.
On a compact, continuous $\Leftrightarrow$ uniformly continuous.
In $\mathbb{R}^n$ Euclidean, totally bounded $\Leftrightarrow$ bounded so compact $\Leftrightarrow$ closed and bounded.
Fails in infinite dimension: balls in $\ell^2(\mathbb{N})$ are not totally bounded, hence not compact.
Characterization of compact sets in $C(X)$, X compact (Arzelà-Ascoli):

compact $\Leftrightarrow$ closed, bounded and equicontinuous.

Lecture 6. Baire Category Theorem: complete metric spaces and locally compact spaces are Baire spaces.
If a Baire space can be covered by a countably many closed F_n, then $\bigcup_n \mathring{F}_n$ is dense.
Fun: the algebraic closure of $\mathbb{Q}_p$ is not a Baire space.

Lecture 7. Normed linear spaces: definition, examples, continuity of operations, products and quotients.
Bases: algebraic bases, topological bases, Schauder bases. A normed linear space with a Schauder basis is separable. The converse holds for Hilbert spaces, not for Banach spaces.
Examples: $\{\delta_n\}$ is a Schauder basis in ℓ^p for $1 < p < \infty$, not for $p = \infty$; $\{x^n\}$ is a Schauder basis of $(C([a,b], \|\cdot\|_\infty))$ by the Weierstrass Approximation Theorem.
Linear maps: equivalent conditions for continuity, basic properties of the operator norm.
In finite dimension, linear maps are bounded. Counterexample: on $c_{00}(\mathbb{N}) \simeq \mathbb{R}[X]$.

Week 3

Lecture 8. If E, F are normed spaces, so is $\mathcal{L}(E, F)$, and it is complete if F is.
What should isomorphisms between normed spaces be?
Example: the map defined on $c_{00}(\mathbb{N})$ by $T\delta_n = \frac{1}{n}\delta_n$ is bounded, but its inverse isn't.
Therefore, we define isomorphisms of normed spaces as linear bicontinuous bijections.
Generalities on Banach spaces: definition, ordinary examples and their duals.
Characterization of Banach spaces: convergence of absolutely convergent series.
Quotients of Banach spaces by closed subspaces are Banach spaces.
Finite dimension: equivalence of norms and corollaries. Riesz Theorem: a normed linear space has finite dimension if and only if the unit ball is compact.
Countable dimension: a normed space with a countable algebraic basis is not complete.
Separation and completion can be performed in a way that preserves the linear structure.
Incomplete spaces such as $c_{00}(\mathbb{N})$, or more generally $C_c(X)$, can be completed into Banach spaces. This gives an alternate construction of L^p-spaces, and a useful dense subspace.
Fun: Ostrowski's Theorem.

Problem session 2. Applications of Arzelà-Ascoli and Baire.
In a normed space, the distance to a finite dimensional subspace is attained.
Compact sets of Hölder maps. A linear space with a countable basis cannot be complete.
If $f \in C([0, +\infty))$ satisfies $\lim_{n\to\infty} f(nx) = 0$ for every $x > 0$, then $\lim_{x\to\infty} f(x) = 0$.
Nowhere differentiable continuous functions are uniformly dense in $C([0, 1])$.
Solutions presented by: Sara, Angelica, Ben.

Lecture 9. Open Mapping Theorem. Consequence: the Bounded Inverse Theorem. If a linear space is complete for two norms N_1 and N_2 such that $N_1 \leq c \cdot N_2$, then $N_1 \sim N_2$.

Lecture 10. Closed Graph Theorem. Unbounded, closable operators (definitions).
A closed subspace V in a Banach space E has a closed linear complement if and only if there is a continuous projection $E \to V$.
Uniform Boundedness Principle (Banach-Steinhaus).

Week 4

Lecture 11. Introduction to duality: functions and distributions, after L. Schwartz.
Hyperplanes and linear functionals: hyperplanes in normed spaces are either dense or closed. Closed hyperplanes are the kernels of bounded linear functionals. $E^* = \mathcal{L}(E, K)$.
Sublinear functionals, analytic Hahn-Banach Theorem.

Lecture 12. Consequences of analytic Hahn-Banach: norm-preserving extensions of bounded linear functionals, separation of points and closed subspaces, characterization of the closure of a subspace by the intersection of closed hyperplanes that contain it.
Fun: Dirac notation: bras and kets.

Problem session 3. Closed subspaces of C^1 functions in $C([0,1])$ are finite-dimensional.
Proof of the Open Mapping Theorem *via* the Bounded Inverse Theorem.
Continuous bilinear forms on Banach spaces.
Solutions presented by: Kyutae, Sam, Ben.

Lecture 13. Gauge of a convex set. Geometric Hahn-Banach. Density criterion.
Fun: a topology can be defined by its closure operation.

Week 5

Lecture 14. Topology induced by a family of semi-norms: neighborhoods, characterizations of continuity, metrizability in the countable case.
Weak topology on a normed linear space, weak convergence. The weak topology is Hausdorff. Convergence implies weak convergence, weakly convergent sequences in a Banach space are bounded. For a convex set, weakly closed $\Leftrightarrow$ closed. Application: non-negativity and norms are preserved by weak limits in L^p-spaces.
Fun: Fréchet topology on $\mathcal{S}(\mathbb{R})$, tempered distributions.

Lecture 15. The map $J : x \longmapsto \text{eval}_x$ is an isometric embedding $E \longrightarrow E^{**}$. A normed linear space is said reflexive if J is surjective.
Weak-$*$ topology on E^{**}, weak-$*$ convergence.
Properties when E Banach: the weak-$*$ topology is Hausdorff, weaker than the weak topology on E^*, weak-$*$ convergent sequences are bounded, weak-$*$ limits of bounded functionals are bounded. Metrizability of the unit ball in (E^*, w^*) when E is separable.
Alaoglu's Theorem: the closed unit ball in E^* is weak-$*$ compact.
Kakutani's Theorem: a Banach space is reflexive iff its closed unit ball is weakly compact (no proof).
Fun: differentiation of tempered distributions, Sobolev spaces.

Midterm 1. $(c_0(\mathbb{N}), \|\cdot\|_\infty)$ is complete, $(C([0,1]), \|\cdot\|_1)$ isn't.
2. Compact operators between Banach spaces, integral operators with continuous kernel.
3. Pointwise limits of continuous maps on a Banach space are continuous on a dense subset.

Lecture 16. Introduction to Banach algebras. Motivation: how to invert $1-a$ in an algebra? Case of a nilpotent in $\mathrm{M}_n(\mathbb{C})$, case of $|a|<1$ in $\mathbb{C}$.
$\rightsquigarrow$ In what algebras can we make sense of power series such as $\sum_{n\geq 0} a^n$?
Definition of Banach algebras. Examples: $\mathbb{C}$, $\mathrm{M}_n(\mathbb{C})$, $\mathcal{L}(E)$ with E Banach, $C(X)$ with X compact, $C_0(X)$ with X locally compact, $L^1(\mathbb{R})$ with convolution, $\ell^1(\mathbb{Z})$ with convolution.
Topological groups. Existence and uniqueness of a Haar measure (no proof).
Examples: $(\mathbb{R},+)$, $(\mathbb{Z},+)$, $(\mathbb{R}_+^\times, \times)$, $\mathbb{R}^\times \ltimes \mathbb{R}$ sitting in $\mathrm{SL}(2,\mathbb{R})$.
Convolution on a locally compact group, $(L^1(G, d\mu_{\mathrm{Haar}}), *)$ is a Banach algebra.

Week 6

Lecture 17. Observe that $C(X)$ with X compact is unital while $C_0(X)$ with X locally compact isn't. If $\tilde{X}$ is the Alexandrov compactification of X, then $C(\tilde{X}) \simeq C_0(X) \oplus \mathbb{C}.1$.
Unitalization of non-unital Banach algebra: $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C}$, unique one with codimension 1.
If $\|1-x\|<1$, then x is invertible, invertibles form an open subset, inversion is continuous.
Spectrum, resolvent of an element. Examples: in $\mathcal{A} = C(X)$, $\mathrm{Sp}_{\mathcal{A}}(f) = f(X)$;
in $\mathcal{A} = \mathrm{M}_n(\mathbb{C})$, $\mathrm{Sp}_{\mathcal{A}}(M) = \{\text{eigenvalues of } M\}$. Working in $\mathbb{C}$ guarantees $\mathrm{Sp}_{\mathcal{A}}(M) \neq \emptyset$.
Interlude: Banach-valued analytic functions, Banach-valued Liouville Theorem.
The spectrum is always non-empty and compact; $\mathrm{Sp}_{\mathcal{A}}(a) \subset D(0, \|a\|)$ and $z \longmapsto (z-a)^{-1}$ is analytic on the resolvent of a.

Lecture 18. Spectral radius, Spectral Radius Formula: $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$.
Gelfand-Mazur Theorem: if every $a \neq 0$ is invertible in $\mathcal{A}$ Banach algebra, then $\mathcal{A} \simeq \mathbb{C}$.
Ideals in Banach algebras: the closure of a proper ideal is proper, maximal ideals are closed, every proper ideal is contained in a maximal one. Quotients of Banach algebras by closed proper ideals are Banach algebras, unitality is preserved.
Maximal ideals in $\mathcal{A}$ are in bijection with non-zero homomorphisms, $\longrightarrow \mathbb{C}$ *via* kernels.
Such homomorphisms satisfy $\|h\| = 1$, so their set $\Sigma_{\mathcal{A}}$ is included in the unit ball of $\mathcal{A}^*$.
The maximal ideal space (or spectrum) $\Sigma_{\mathcal{A}}$ is w-$*$ compact and $\mathrm{Sp}_{\mathcal{A}}(a) = \{h(a), h \in \Sigma_{\mathcal{A}}\}$.
Case of $\mathcal{A} = C(K)$: the map $x \mapsto \delta_x$ gives a homeomorphism between K and $\Sigma_{C(K)}$.

Problem session 4. Discussion of the midterm, duality.
Weakly continuous linear maps between Banach spaces are continuous.
'Optimality' of Hahn-Banach: closed convex subsets in $\ell^1(\mathbb{N})$ that cannot be separated.
The dual of $\ell^p(\mathbb{N})$ is $\ell^q(\mathbb{N})$. Use of the Hölder Inequality, finitely supported sequences.
The inclusion $\ell^1(\mathbb{N}) \subset (\ell^\infty(\mathbb{N}))^*$ is strict.
Solutions presented by: Melanie.

Lecture 19. The Gelfand Transform $\mathscr{G} : a \mapsto \mathrm{eval}_a$ is a continuous algebra homomorphism $\mathcal{A} \longrightarrow C_0(\Sigma_{\mathcal{A}})$ such that $\|\mathscr{G}(a)\|_\infty = r(a)$ and its kernel is the radical of $\mathcal{A}$:

$$\mathrm{Rad}(\mathcal{A}) = \bigcap_{\mathcal{J} \text{ max. ideal}} \mathcal{J}.$$

The Gelfand Transform is $\mathscr{G} : \mathcal{A} \longrightarrow C_0(\Sigma_{\mathcal{A}})$ is an isomorphism if $\mathcal{A}$ is an algebra of functions. What about other (abelian) cases?
Example: heuristic determination of $\Sigma_{\mathcal{A}}$ for $\mathcal{A} = (L^1(\mathbb{R}), *)$: functionals of the form $h_\xi : f \mapsto \int_{\mathbb{R}} f(x)e^{i\xi x}dx$ are in $\Sigma_{\mathcal{A}}$. The map $\xi \mapsto h_\xi$ is actually surjective and the Gelfand Transform on $L^1(\mathbb{R})$ is the Fourier transform $L^1(\mathbb{R}_x) \longrightarrow C_0(\mathbb{R}_\xi)$. What about G LCA?
Abstract C*-algebras: definition. It generalizes $(\mathbb{C}, +, \times)$ with the complex conjugation.
Examples: $\mathrm{M}_n(\mathbb{C})$ with $A^* = {}^t\overline{A}$, $C_0(X)$ with $f^*(x) = \overline{f(x)}$.
Non-example: $(L^1(G), *)$ for G unimodular with $f^*(g) = \overline{f(g^{-1})}$ is an involutive Banach algebra, but the C*-condition $\|a^*a\| = \|a\|^2$ is not satisfied. (Counter-example!)

Week 7

Lecture 20. Properties of C*-algebras: the involution is isometric, hermitian, normal and uitary elements. Every a in $\mathcal{A}$ has a unique expression $a = a_1 + ia_2$ with a_1, a_2 hermitian and a is normal iff they commute.
Unitalization: $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C}$, spectrum in non-unital algebras: $\mathrm{Sp}'_{\mathcal{A}}(a) = \mathrm{Sp}_{\mathcal{A}_1}(a)$.
Morphisms of C*-algebras: $*$-morphisms, no continuity assumption.
If a is unitary, then $\|a\| = 1$; if a is hermitian, then $\|a\| = r(a)$. Corollaries: morphisms of C*-algebras are automatically continuous, with norm ≤ 1; in a C*-algebra, the norm is determined by the spectral radius: $\|a\| = \sqrt{r(a^*a)}$ for all a in $\mathcal{A}$.
Hermitian elements have real spectrum.

Lecture 21. Gelfand-Naimark Theorem: a commutative C*-algebra $\mathcal{A}$ is isometrically isomorphic to $C_0(\Sigma_{\mathcal{A}})$ *via* the Gelfand Transform (Using Stone-Weierstrass without proof).
Continuous functional calculus in C*-algebras (no proof). Properties: if f is real-valued then $f(a)$ hermitian; if f is $\mathbb{T}$-valued, then $f(a)$ unitary.
Spectral Mapping Theorem: $\mathrm{Sp}(f(a)) = f(\mathrm{Sp}(a))$.
Fun: what are C*-algebras good for? Overview of NCG, irrational foliation of $\mathbb{T}^2$.

Problem session 5. Let E be a Banach space. A sequence in E^* is weakly convergent if and only if it is bounded and converges pointwise on a dense subset.
Example of a weakly convergent but not convergent sequence: $\sin(n\cdot)$ in $L^2([a, b])$.
Some properties of the inclusion $L^1(\mathbb{R}) \subset C_0(\mathbb{R})^*$.
Solutions presented by: Angelica, Sam.

Lecture 22. Inner products, examples, Cauchy-Schwarz Inequality, polarization identity, Hilbert space, completion of a pre-Hilbert space.
Orthogonality, orthogonal of a subset, Pythagorean Theorem, Parallelogram Law.
Distance to a closed convex, orthogonal projection onto a closed subspace.
Fun: let $G \curvearrowright X$ be a group action such that all the orbits are infinite. Prove that G separates the finite subsets of X.

Week 8

Lecture 23. Properties of the orthogonal projection $p_{\mathcal{K}}$ on a closed subspace $\mathcal{K}$ in a Hilbert $\mathcal{H}$: linearity, idempotence, kernel and range. Corollaries: $p_{\mathcal{K}^\perp} = 1 - p_{\mathcal{K}}$, $(\mathcal{K}^\perp)^\perp = \mathcal{K}$, orthogonal complement: $\mathcal{K} \oplus \mathcal{K}^\perp = \mathcal{H}$. For any subset $A \subset \mathcal{H}$, $(A^\perp)^\perp = \overline{\operatorname{span}}(A)$.
Density criterion: a linear subspace is dense if and only if its orthogonal is $\{0\}$.
Riesz Representation Theorem: bounded linear functionals on a Hilbert space are of the form $x \mapsto \langle \cdot, x_0 \rangle$. This gives a conjugate linear isometry $\mathcal{H} \simeq \mathcal{H}^*$.
Corollary: norm-preserving extensions of bounded linear functionals.
Hilbert bases: maximal orthonormal families. Existence by Zorn. In a separable Hilbert space, orthonormal families are countable. For $\mathcal{F} = \{e_n\}_{n\geq 1}$ orthonormal sequence, $\sum_{n\geq 1} |\langle x, e_n\rangle|^2 \leq \|x\|^2$ (Bessel's Inequality) and $\sum_{n\geq 1}\langle x, e_n\rangle e_n$ converges to the projection of x on $\overline{\operatorname{span}}(\mathcal{F})$. Characterization of Hilbert bases, including Parseval's Identity $\sum_{n\geq 1} |\langle x, e_n\rangle|^2 = \|x\|^2$.

Lecture 24. Linear maps between inner-product spaces preserve norms if and only if they preserve inner-products. Unitary isomorphisms.
Example of non-surjective isometric embedding: the unilateral shift on $\ell^2(\mathbb{N})$.
All separable Hilbert spaces are isometrically isomorphic. Example: if $\nu << \mu$ on X and $\varphi = \frac{d\mu}{d\nu}$ is the Radon-Nykodim derivative, then $L^2(X, \mu) \simeq L^2(X, \nu)$ *via* $f \mapsto \sqrt{\varphi} f$.
Bounded sesquilinear forms are of the form $u(x_1, x_2) = \langle Ax_1, x_2\rangle_2 = \langle x_1, Bx_2\rangle_1$.
The operators A and B are the adjoint of each other. Properties of $A \mapsto A^*$.
Characterisation of isometric embeddings and isometries in terms of adjoints (no proof).
Exercise: for A in $\mathcal{B}(\mathcal{H})$, $\ker A = (\operatorname{ran} A^*)^\perp$ and $\ker A^* = (\operatorname{ran} A)^\perp$.
A bounded operator A on $\mathcal{H}$ is self-adjoint if and only if $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$.
If $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then $\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|$.
Consequence: $\mathcal{B}(\mathcal{H})$ is a C*-algebra. So is every closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.
Gelfand-Naimark-Segal Theorem: every C*-algebra arises in this way. *C*-algebras are operator algebras.*

Problem session 6. Positive elements in C*-algebras.
Elements of noncommutative topology: unitality, ideals, quotients, connected components.
Solutions presented by: Melanie, Sara.

Lecture 25. Fourier series: motivation (wave equation), definitions.
L^2-theory: $\{e^{in\cdot}\}_{n\in\mathbb{Z}}$ is a Hilbert basis of $L^2([0, 2\pi], \frac{\lambda}{2\pi})$.
Consequences: Bessel, Parseval, mean-square convergence.
Overview of known results: Dirichlet, Fejér, Carleson, Hunt.
Dirichlet and Fejér method: partial sums of Fourier series are given by good kernels.

Week 9

Lecture 26. Observation: $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$.
Unitary representations of locally compact groups. Examples: standard representation of

U(n), characters χ_n of $\mathbb{R}/2\pi\mathbb{Z}$, left regular representation λ_G, trivial representation.
Orthogonal complements, irreducible representations. Intertwiners, unitary equivalences.
Schur's Lemma (finite dimensional proof). A representation is irreducible if and only if it has no non-scalar self-intertwiners (finite dimensional proof).
Corollary: irreducible representations of abelian groups are 1-dimensional.
If G is locally compact abelian, $\hat{G}$ is the set of unitary characters, *i.e.* continuous multiplicative maps $\chi : G \longrightarrow \mathbb{T}$. Determination of $\hat{\mathbb{R}}$ and $\hat{\mathbb{T}}$. Fourier Transform on $L^1(G)$.

Problem session 7. Gram-Schmidt orthonormalization, orthogonal polynomials, a group acting on a discrete set with infinite orbits separates finite subsets.
Solutions presented by: Ben, Angelica.

Lecture 27. Plancherel formula for locally compact abelian groups, Fourier inversion: $f(g) = \displaystyle\int_{\hat{G}} (\chi * f)(g)\, d\chi$ for 'good' functions on G. Generalization to compact (Peter-Weyl) and reductive groups (Harish-Chandra).
Unitary representations of G and non-degenerate representations of $L^1(G)$, the reduced norm. Reduced C*-algebra of locally compact groups. Abelian case: determination of the spectrum of $L^1(G)$, relation between Gelfand and Fourier transformations.
For G locally compact abelian group, $\mathrm{C}^*_r(G) = C_0(\hat{G})$.

Problem session 8. (Homework) Pointwise and uniform convergence of Fourier series under regularity assumptions: (piecewise) continuity, piecewise C^1 class. Application to the determination of sums of series. Existence of divergent Fourier series as a consequence of The Uniform Boundedness Principle.

- End of the course -

Problem set 1: Metric spaces

1. Distance to a subset and metric Urysohn's Lemma

Let (E, d) be a metric space. For any subset $A \subset E$ and any point $x \in E$, the *distance* between x and A is defined by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

(1) **Show that $d(x, A) = d(x, \bar{A})$.**

(2) **Show that $d(\cdot, A)$ is 1-Lipschitz.**

(3) **Let A and B be disjoint closed subsets of E. Prove the existence of a continuous function $f : E \longrightarrow \mathbb{R}$ such that:**
 (a) $0 \leq f(x) \leq 1$ **for all $x \in E$;**
 (b) $f(x) = 0$ **for all $x \in A$;**
 (c) $f(x) = 1$ **for all $x \in B$.**

Solution.

(1) Observe that $A \subset \bar{A}$ so $d(x, A) \geq d(x, \bar{A})$. For the other inequality, consider α in $\bar{A}$. There exists a sequence $\{a_n\} \in A^{\mathbb{N}}$ that converges to α. Given x fixed, the function $d(x, \cdot)$ is continuous so $\lim_{n\to\infty} d(x, a_n) = d(x, \alpha)$. Since $d(x, a_n) \geq d(x, A)$ for every n, it follows that $d(x, \alpha) \geq d(x, A)$. This is true for every α in $\bar{A}$ so $d(x, \bar{A}) \geq d(x, A)$.

(2) For $x, y \in E$ and $a \in A$, the triangle inequality and the definition of $d(x, A)$ imply that $d(x, A) \leq d(x, y) + d(y, a)$. This is true for every $a \in A$ so $d(x, A) \leq d(x, y) + d(y, A)$ and we get $d(x, A) - d(y, A) \leq d(x, y)$. The same argument gives $d(y, A) - d(x, A) \leq d(x, y)$ hence the result.

(3) Consider $x \longmapsto \dfrac{d(x, A)}{d(x, A) + d(x, B)}$.

Problem 2. Completeness is not a topological property

Let $E = (0, +\infty)$ and for $x, y \in E$, consider $\delta(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$.

(1) **Prove that δ is a distance on E and that it induces the same topology as the Euclidean distance d.**

(2) **Is the map $x \mapsto x^{-1}$ uniformly continuous between (E, d) to itself ? As a map from (E, d) to (E, δ)?**

(3) **Is (E, δ) complete ? What about $((0, 1], d)$ and $((0, 1], \delta)$?**

Solution.

(1) *Method 1*: prove that every open d-ball contains a δ-ball with the same center and vice versa.
Method 2: prove that $(E, d) \xrightarrow{\mathrm{Id}} (E, \delta)$ is a homeomorphism. To see this, it is convenient to decompose the identity map as $(E, d) \xrightarrow{\varphi} (E, d) \xrightarrow{\varphi} (E, \delta)$ where $\varphi(x) = x^{-1}$ and prove that both are homeomorphisms. Note that both methods boil down to the fact that φ is a homeomorphism from $(E, d \,\mathrm{or}\, \delta)$ to $(E, d \,\mathrm{or}\, \delta)$.

(2) No. Yes.

(3) No: $u_n = n$ is Cauchy but it does not converge (argue by contradiction). No: it is not closed in $(\mathbb{R}, d)$. Alternatively, consider $u_n = \frac{1}{n}$, Cauchy but not convergent in $(0, 1]$.
Yes. *Method 1*: show that a Cauchy sequence $\{u_n\}$ for δ is also Cauchy for d hence converges for d in the closure of $(0, 1]$. If the d-limit is > 0, it is also the δ-limit because d and δ induce the same topology (or check it directly with balls) so the sequence converges. Assume the limit is 0. Then $\delta(1, u_n)$ diverges to $+\infty$ so $\{u_n\}$ is not bounded which is impossible since it is Cauchy. *Method 2*: $x \mapsto x^{-1}$ is an isometry (hence uniformly continuous) between $((0, 1], \delta)$ and $([1, +\infty), d)$, which is closed in $(\mathbb{R}, d)$ complete, so is complete.

Problem 3. The Banach Contraction Principle

Let (E,d) be a complete metric space and $f : E \longrightarrow E$.

(1) **Show that if f is k-Lipschitz with $k < 1$, the equation $f(x) = x$ has a unique solution in E.**

(2) **Show that if E is compact, it is enough to have $d(f(x), f(y)) < d(x,y)$ for all x, y to obtain the same result.**

Solution.

(1) Think triangle inequality and geometric series.

(2) The real-valued function $x \mapsto d(x, f(x))$ is continuous on a compact set so it is bounded and the extrema are attained.

Problem 4. Completeness of $\ell^2(\mathbb{N})$

Show that the set of sequences $U = \{u_n\}$ such that $\sum_{n\geq 0} |u_n|^2$ converges is complete for the norm $\|U\|_2 = \left(\sum_{n=0}^{\infty} |u_n|^2\right)^{\frac{1}{2}}$.

Solution.
The skeleton of the proof we studied for the space of bounded functions with values in a complete space carries over.

Problem 5. Cantor's Intersection Theorem

Let (E,d) be a metric space and $A \subset E$ a non-empty subset. The *diameter* of A is defined by

$$\operatorname{diam}(A) = \sup_{x,y\in A} d(x,y).$$

Prove that E is complete if and only if for every decreasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed subsets of E such that $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$, there is a point x such that

$$\bigcap_{n\in\mathbb{N}} F_n = \{x\}.$$

Solution.
See Section 9.4 of [Royden-Fitzpatrick].

Problem 6. Characterizations of compactness for metric spaces

Let (E, d) be a metric space. Prove that the following conditions are equivalent.

(1) E **has the *Borel-Lebesgue* property, *i.e.* is topologically compact.**

(2) **If $\mathcal{F}$ is a family of closed subsets of E such that every subfamily has nonempty intersection, then $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$.**

(3) E **is complete and *totally bounded i.e.* can be covered by finitely many open balls of radius ε, for any $\varepsilon > 0$.**

(4) E **has the *Bolzano-Weierstrass* property, *i.e.* is sequentially compact.**

Solution.
The equivalence between (i) and (ii) holds in topological (non-necessarily metric) spaces. See Propositions 17, 18 and 19 in Section 9.5 of [Royden-Fitzpatrick] for the rest.

Problem set 2: Applications of the Arzelà-Ascoli and the Baire Theorems

Problem 1. Hölder maps

A function $f \in C([0,1],\mathbb{R})$ is said to be α-*Hölder* if

$$h_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is finite. For $M > 0$ and $0 < \alpha \leq 1$, denote

$$H_{\alpha,M} = \{f \in C([0,1],\mathbb{R})\, ,\, h_\alpha(f) \leq M \, \textbf{and} \, \|f\|_\infty \leq M\}\, .$$

Prove that $H_{\alpha,M}$ is compact in $(C([0,1],\mathbb{R}), \|\cdot\|_\infty)$.

Solution.
The Arzelà-Ascoli Theorem implies that it suffices to check that $H_{\alpha,M}$ is closed, bounded and equicontinuous. The set in question is the intersection of the closed ball $B_c(0,M)$ and $F = \{f \in C([0,1])\, ,\, h_\alpha(f) \leq M\}$, so it is automatically bounded and it is enough to check that F is closed. To do so, consider a sequence $\{f_n\}$ of functions in F, that converges to f in $C([0,1])$. The pointwise convergence of the sequence implies that $\frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq M$ for every $x \neq y$ so F is closed. To establish equicontinuity, let $\varepsilon > 0$ and verify that $\delta = \left(\frac{\varepsilon}{M}\right)^{\frac{1}{\alpha}}$ is an appropriate modulus of continuity.

Problem 2. Show that a normed linear space over $\mathbb{R}$ that has a countable algebraic basis cannot be complete.

Solution.
Let E be a normed space with an algebraic basis $\{e_i\}_{i\in\mathbb{N}}$ and $F_n = \mathrm{span}(e_1,\ldots,e_n)$. Each F_n is finite-dimensional, hence closed. Moreover, if F_n contained an open ball of radius $r > 0$ it would also contain $B(0,r)$, which generates E, so E would be contained in F_n. Therefore, each F_n has empty interior and Baire's Theorem ensures that $\bigcup_{n\geq 1} F_n$ has empty interior too, which contradicts the fact that $\bigcup_{n\geq 1} F_n = E$.

Problem 3. Let $f : [0,+\infty) \longrightarrow \mathbb{R}$ be continuous and assume that for all $x > 0$,

$$\lim_{n\to\infty} f(nx) = 0.$$

Prove that $\lim\limits_{x\to\infty} f(x) = 0$.

Hint: for $\varepsilon > 0$ and $n \in \mathbb{N}$, consider $F_{n,\varepsilon} = \{x \geq 0\, ,\, \forall p \geq n\, ,\, |f(px)| \leq \varepsilon\}$.

Solution.
Each $F_{n,\varepsilon}$ is closed as the intersection of inverse images of the closed subset $[0,\varepsilon]$ of $\mathbb{R}$

by the continuous functions $f(p\,\cdot)$ for $p \in \mathbb{N}$, $p \geq n$. The hypothesis on f implies that $(0,+\infty) \subset \bigcup_{n\geq 1} F_n$. Being locally compact, $(0,+\infty)$ is a Baire space so that there exists $n_0 \in \mathbb{N}$ such that $\overset{\circ}{F}_{n_0} \neq \emptyset$.
In other words, there exist $0 < \alpha < \beta$ such that $(\alpha, \beta) \subset F_{n_0}$, which means that

$$\forall x \in (\alpha, \beta)\,,\, \forall p \geq n_0 \quad , \quad |f(px)| \leq \varepsilon.$$

The result then follows from the fact that, for p large enough, the intervals $(p\alpha, p\beta)$ overlap. More precisely, the condition $(p+1)\alpha < p\beta$ is equivalent to $p > \frac{\alpha}{\beta-\alpha}$ so that if $N > \max(n_0, \frac{\alpha}{\beta-\alpha})$, one has $|f(x)| \leq \varepsilon$ for x in $\bigcup\limits_{p\geq N}(p\alpha, p\beta) = (N\alpha, +\infty)$.

Problem 4. Show that nowhere differentiable functions are dense in $E = C([0,1],\mathbb{R})$ equipped with its ordinary norm.

Hint: consider, for $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$U_{n,\varepsilon} = \left\{ f \in E\,,\, \forall x \in [0,1]\,,\, \exists y \in [0,1]\,,\, |x-y| < \varepsilon \quad \text{and} \quad \left| \frac{f(y)-f(x)}{y-x} \right| > n \right\}.$$

Solution.
We first prove that each set $U_{n,\varepsilon}$ is open because its complement $U^c_{n,\varepsilon}$ is closed. Observe that

$$U^c_{n,\varepsilon} = \left\{ f \in E\,,\, \exists x \in [0,1]\,,\, \forall y \in [0,1]\,,\, |x-y| < \varepsilon \Rightarrow \left| \frac{f(y)-f(x)}{y-x} \right| \leq n \right\}.$$

and let $\{f_k\}$ be a sequence in $U^c_{n,\varepsilon}$ that converges to f in E. For each k, there exists $x_k \in [0,1]$ such that $|x_k - y| < \varepsilon \Rightarrow \left|\frac{f(y)-f(x_k)}{y-x_k}\right| \leq n$. Since $[0,1]$ is compact, $\{x_k\}$ has a convergent subsequence $\{x_{\varphi(k)}\}$. Denote x its limit and let y in $[0,1]$ be such that $0 < |x-y| < \varepsilon$. For k large enough, one has $0 < |x_{\varphi(k)} - y| < \varepsilon$ so that $\left|\frac{f_{\varphi(k)}(y)-f_{\varphi(k)}(x_{\varphi(k)})}{y-x_{\varphi(k)}}\right| \leq n$ and the uniform convergence $f_{\varphi(k)} \to f$ implies that $\left| \dfrac{f(y)-f(x)}{y-x} \right| \leq n$, so that f belongs to $U^c_{n,\varepsilon}$.

Now we prove that $U_{n,\varepsilon}$ is dense in E. Polynomials are dense in E, so it suffices to prove that functions of class C^1 can be approximated by elements of $U_{n,\varepsilon}$.
For $p \geq 1$ integer, let v_p be a continuous function on $[0,1]$, affine on each interval $\left[\dfrac{k}{2p}, \dfrac{k+1}{2p}\right]$ and such that $v_p\left(\frac{k}{2p}\right) = 0$ (resp. $= 1$) if k is even (resp. odd). Let f be a function of class C^1 on $[0,1]$ and $g_p = f + \lambda v_p$. By construction, $\|f - g_p\|_\infty \leq \lambda$ so g_p can be chosen arbitrarily close to f.

If $x \neq y$ in $[0,1]$, then

$$\begin{aligned}\left|\frac{g_p(x) - g_p(y)}{x-y}\right| &\geq \lambda \left|\frac{v_p(x) - v_p(y)}{x-y}\right| - \left|\frac{f(x) - f(y)}{x-y}\right| \\ &\geq \lambda \left|\frac{v_p(x) - v_p(y)}{x-y}\right| - \|f'\|_\infty.\end{aligned}$$

Let $p > \dfrac{1}{2\lambda}(n + \|f'\|_\infty)$. For any $x \in [0,1]$, there exists $y \in [0,1]$ within ε of x and in the same interval $\left[\frac{k}{2p}, \frac{k+1}{2p}\right]$. By definition of v_p, the latter implies that $\left|\frac{v_p(x)-v_p(y)}{x-y}\right| = 2p$. Then

$$\left|\frac{g_p(x) - g_p(y)}{x-y}\right| \geq 2p\lambda - \|f'\|_\infty > n$$

so that $g_p \in U_{n,\varepsilon}$.

The Baire Category Theorem ensures that $U = \bigcap_{n\geq 1} U_{\frac{1}{n},n}$ is dense in E. Let $f \in U$ and $x \in [0,1]$. Then there is a sequence $\{x_n\}$ such that $0 < |x_n - x| < \frac{1}{n}$ and $\left|\frac{f(x_n)-f(y)}{x_n-y}\right| > n$, which prevents f from being differentiable at x.

Problem set 3: Linear operators on Banach spaces

Problem 1. Let E be the space $C([0,1])$ equipped with $\|\cdot\|_\infty$. If f is differentiable, we write $D(f) = f'$.

(1) **Let F be a closed subspace of E that is included in $C^1([0,1])$.**
 (a) **Show that $D : F \longrightarrow E$ is Lipschitz.**
 (b) **Prove that F is finite dimensional.**

(2) **Let $G = (C^1([0,1]), \|\cdot\|_\infty)$.**
 (a) **Show that $D : G \longrightarrow E$ is closed.**
 (b) **Is it continuous?**

Hints: 1.(a) Study the graph of D. - 1.(b) Study the unit ball of F.

Solution.

(1) (a) Both E and F are Banach spaces and D is linear so by the Closed Graph Theorem, it suffices to prove that D is closed. Let $\{f_n\}$ be a sequence in F such that f_n and $Df_n = f_n'$ converge, say to f and g respectively. This means that $f_n \to f$ and $f_n' \to g$ uniformly, which implies that $f' = g$, so that (f, g) belongs to the graph of D, which is therefore continuous, hence Lipschitz.

 (b) By Riesz's Theorem, it is enough to prove that the closed unit ball in F is compact. It is bounded by definition and closed in $E = C([0,1])$ so, it suffices to prove that it is equicontinuous. Let C be a Lipschitz constant for D. Then $\|f'\|_\infty \leq C\|f\|_\infty \leq C$ for all f in the unit ball of F, which is therefore uniformly equicontinuous.

(2) (a) This was already done in 1.(a): take $F = G$.

 (b) No: consider the sequence $\{f_n : x \longmapsto x^n\}$ in the unit ball of G and its image under D. Note that this does not contradict the Closed Graph Theorem, since G is not closed in E and therefore not complete.

Problem 2. Let E be a normed linear space, F a closed subspace of E and

$$\pi : E \longrightarrow E/F$$

the natural surjection.

(1) **Let $x \in E$ and $r > 0$. Show that $\pi(B(x,r)) = B(\pi(x), r)$.**

(2) **Let $\mathcal{U} \subset E/F$. Prove that $\mathcal{U}$ is open if and only if $\pi^{-1}(\mathcal{U})$ is open in E.**

(3) **Prove that π is an open map.**

(4) **Derive the Open Mapping Theorem from the Bounded Inverse Theorem.**

Solution.

(1) First, observe that translations are isometries that commute to π so we may assume $x = 0$. The surjection π is 1-Lipschitz by definition of the norm on E/F so $\pi(B(0,r)) \subset B(\pi(0), r)$.
Conversely, assume that $y \in E/F$ has norm $< r$. Choose $x \in E$ such that $\pi(x) = y$. Then $\|y\| = \inf_{v \in F} \|x + v\| < r$ so there exists $v \in F$ such that $\|x + v\| < r$ and y has a preimage in $B(0,r)$, which means that $B(\pi(0), r) \subset \pi(B(0,r))$.

(2) Again, π being 1-Lipschitz, it is continuous, which implies that if $\mathcal{U}$ is open in E/F, then $\pi^{-1}(\mathcal{U})$ is open in E.
For the converse, assume that $\pi^{-1}(\mathcal{U})$ is open in E and let $y \in \mathcal{U}$, with preimage $x \in E$. Since $\pi^{-1}(\mathcal{U})$ is open and contains x, there exists $r > 0$ such that $B(x,r) \subset \pi^{-1}(\mathcal{U})$.
By the result of the previous question, $\mathcal{U} = \pi(\pi^{-1}(\mathcal{U}))$ contains the ball $B(y,r)$ so it is a neighborhood of y.

(3) Let U be open in F. By the result of the previous question, in order to prove that $\pi(U)$ is open in E/F, it suffices to prove that $\pi^{-1}(\pi(U))$ is open in E, which follows from the observation that $\pi^{-1}(\pi(U)) = U + F = \bigcup_{v \in F} U + v$.

(4) Let $T : E \longrightarrow F$ be a surjective continuous linear map between Banach spaces. Consider the induced map $\tilde{T} : E/\ker T \longrightarrow F$. Apply the Bounded Inverse Theorem to $\tilde{T}$ and conclude by noticing that $T = \tilde{T} \circ \pi$.

Problem 3. Bilinear maps. Let E_1, E_2 and F be normed linear spaces and equip $E_1 \times E_2$ with the norm $\|(x,y)\| = \max(\|x\|, \|y\|)$. A map $B : E_1 \times E_2 \longrightarrow F$ is said ***bilinear*** if all the maps

$$\begin{array}{rcl} \Lambda_x : E_2 & \longrightarrow & F \\ y & \longmapsto & B(x,y) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathrm{P}_y : E_1 & \longrightarrow & F \\ x & \longmapsto & B(x,y) \end{array}$$

are linear. Moreover, B is said

- ***separately continuous*** if all the maps Λ_x and P_y are continuous;
- ***bounded*** if

$$\|B\| := \sup\{\|B(x,y)\| \quad , \quad x \in E_1 \, , \, y \in E_2 \, , \, \|x\| \le 1 \, , \, \|y\| \le 1\} < \infty.$$

(1) Show that the statements
  (a) B is bounded.
  (b) There exists a constant $C \ge 0$ such that $\|B(x,y)\| \le C\|x\|\|y\|$ for all (x,y) in $E_1 \times E_2$.
  (c) B is continuous.
  (d) B is continuous at $(0,0)$.

  are equivalent and that if they hold, $\|B\|$ is the smallest C in (b).

Recall that the set of bounded linear maps between linear spaces E and F is denoted by $\mathcal{L}(E,F)$. The set of bounded bilinear maps from $E_1 \times E_2$ to F will be denoted by $\mathcal{B}(E_1 \times E_2, F)$.

(2) Let E and F be normed linear spaces. Show that the map

$$\begin{array}{rcl} \beta : \mathcal{L}(E,F) \times E & \longrightarrow & F \\ (T,x) & \longmapsto & T(x) \end{array}$$

is in $\mathcal{B}\left(\mathcal{L}(E,F) \times E, F\right)$ and that $\|\beta\| \le 1$.

(3) Let E, F and G be normed linear spaces. Show that the map

$$\begin{array}{rcl} \gamma : \mathcal{L}(F,G) \times \mathcal{L}(E,F) & \longrightarrow & \mathcal{L}(E,G) \\ (S,T) & \longmapsto & S \circ T \end{array}$$

is in $\mathcal{B}\left(\mathcal{L}(F,G) \times \mathcal{L}(E,F), \mathcal{L}(E,G)\right)$ and that $\|\gamma\| \le 1$.

(4) Show that $\mathcal{B}(E_1 \times E_2, F)$ equipped with the pointwise operations and $\|\cdot\|$ defined above is a normed linear space.

(5) (a) Show that $\mathcal{B}(E_1 \times E_2, F)$ is isometrically isomorphic to $\mathcal{L}\left(E_1, \mathcal{L}(E_2, F)\right)$.
  (b) What can be said of $\mathcal{B}(E_1 \times E_2, F)$ if F is a Banach space?

(6) Assume that E_1 and E_2 are Banach spaces. Show that a bilinear map $B : E_1 \times E_2 \longrightarrow F$ is bounded if and only if it is separately continuous.

(7) **Consider** $E = \mathbb{R}[X]$ **equipped with the norm** $\|P\| = \int_0^1 |\tilde{P}(x)|\, dx$ **where** $\tilde{P}$ **is the function associated with the polynomial** P**. Show that the bilinear map** α **defined on** $E \times E$ **by** $\alpha(P, Q) = \int_0^1 \tilde{P}(x)\tilde{Q}(x)\, dx$ **is separately continuous but not bounded.**

Solution.
The equivalences in 1. and the statement in 4. can be proved in the same fashion as the analogous ones in the case of linear maps. The results in 2. and 3. are direct consequences of the properties of the operator norm.

(5) (a) Consider the map $x \longmapsto \Lambda_x$.

(b) It is a Banach space.

(6) The implication (bounded $\Rightarrow$ separately continuous) is trivial. Conversely, assume B separately continuous and fix x in E_1 with $\|x\| \leq 1$. Then $|\Lambda_x(y)| = |B(x, y)| \leq \|\mathrm{P}_y\|$ for all y in E_2. By the Uniform Boundedness Principle, the family $\{\Lambda_x\ ,\ \|x\| \leq 1\}$ is bounded in $\mathcal{L}(E_2, \mathbb{R})$ so there exists a constant C such that $|B(x, y)| = |\Lambda_x(y)| \leq C$ for all x, y in the closed unit ball of $E_1 \times E_2$. Note that it suffices to assume only one of the E_i to be complete for the argument to work.

(7) Separate continuity follows from the fact that $\|\Lambda_P\| \leq \|P\|_\infty$ and the symmetry of α. For $n \geq 1$, the polynomial nX^n lies on the unit sphere of E and $\alpha(nX^n, nX^n) = \dfrac{n^2}{2n+1} \underset{n\to\infty}{\longrightarrow} +\infty$ so α is not bounded.

Problem set 4: Duality

Problem 1. Let E and F be Banach spaces and $T : E \longrightarrow F$ a linear map such that

$$\forall \varphi \in F^* \quad , \quad \varphi \circ T \in E^*.$$

Prove that T is bounded.

Solution.
By the Closed Graph Theorem, it is enough to consider a sequence $\{(x_n, Tx_n)\}_{n\in\mathbb{N}}$ that converges to (x, y) in $E \times F$ and to prove that $Tx = y$. Let $\varphi \in F^*$. The continuity of φ and that of $\varphi \circ T$ imply that $\varphi(Tx) = \varphi(y)$ for all $\varphi \in F^*$. By Hahn-Banach, this implies that $Tx = y$.

Problem 2. Closed convex sets that cannot be separated.
Let E_0 and F be the subsets of $\ell^1(\mathbb{N})$ defined by

$$E_0 = \left\{u \in \ell^1(\mathbb{N})\,,\, \forall n \geq 0\,,\, u_{2n} = 0\right\}$$

and

$$F = \left\{u \in \ell^1(\mathbb{N})\,,\, \forall n \geq 1\,,\, u_{2n} = 2^{-n} u_{2n-1}\right\}.$$

(1) **Verify that E_0 and F are closed subspaces and that $\overline{E_0 + F} = \ell^1(\mathbb{N})$.**

(2) **Let v be the sequence defined by $v_{2n} = 2^{-n}$ and $v_{2n-1} = 0$.**

   (a) **Verify that v is in $\ell^1(\mathbb{N})$ and that $v \notin E_0 + F$.**

   (b) **Let $E = E_0 - v$. Prove that E and F are closed disjoint convex subsets of $\ell^1(\mathbb{N})$ that cannot be separated in the sense that there exists no couple $(\varphi, \alpha) \in (\ell^1(\mathbb{N}))^* \times \mathbb{R}$ such that $\varphi \neq 0$ and**

$$\varphi(e) \leq \alpha \leq \varphi(f)$$

   for all $e \in E$, $f \in F$.

Hints: 1. Finitely supported sequences are dense in $\ell^1(\mathbb{N})$. (See 4.(a) in Prob. 3.)
2.(b) What can be said of a functional that remains bounded on a linear subspace?

Solution.

(1) Both spaces can be written as intersections of kernels of bounded linear functionals so they are closed. To prove the density of the sum, it suffices to prove that it contains all finitely supported sequences. Let u be a sequence supported on $\{0, 1, \ldots, p\}$, that is, $u_n = 0$ for $n > p$. Then let e and f be defined by $\begin{cases} e_{2n} = 0 & e_{2n-1} = u_{2n-1} - 2^n u_{2n} \\ f_{2n} = u_{2n} & f_{2n-1} = 2^n u_{2n} \end{cases}$ and observe that $u = e + f$.

(2) (a) Assume that $v = e + f$ with $e \in E_0$ and $f \in F$. Then, $e_{2n-1} = -1$ and $f_{2n-1} = 1$ for all $n \geq 1$ which contradicts e and f being in $\ell^1(\mathbb{N})$.

(b) If there was a separating couple (φ, α), then φ would be bounded above by $\alpha + \varphi(v)$ on E_0 and bounded below by α on F. Since they are linear subspaces, it means that φ would vanish on both hence on the sum. Since the latter is dense in $\ell^1(\mathbb{N})$ and φ assumed continuous, this implies $\varphi = 0$.

Problem 3. Dual of $\ell^p(\mathbb{N})$.

In this problem, we assume the sequences to be real-valued. Let $p \in [1, +\infty)$ and denote by q the only element of $(1, +\infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The purpose of this problem is to identify $\ell^p(\mathbb{N})^*$ with $\ell^q(\mathbb{N})$.
To this end, we shall prove that the map Φ defined on ℓ^q by

$$\Phi(u)v = \sum_{n \geq 0} u_n v_n$$

is an isometry.

(1) **Verify that $\Phi(u)$ is a linear functional on $\ell^p(\mathbb{N})$ for each $u \in \ell^q(\mathbb{N})$ and that Φ is linear.**

(2) **Let $u \in \ell^q(\mathbb{N})$. Prove that $\Phi(u) \in \ell^p(\mathbb{N})^*$ and that $\|\Phi(u)\| \leq \|u\|_q$.**

(3) **Let $u \in \ell^q(\mathbb{N})$ be fixed.**

(a) **Assume $p > 1$. Verify that the sequence v defined by**

$$v_n = \|u\|_q^{1-q} \operatorname{sign}(u_n) \, |u_n|^{q-1}$$

is in ℓ^p and compute $\Phi(u)v$.

(b) **Let $p = 1$. For $\varepsilon > 0$, find v on the unit sphere of $\ell^1(\mathbb{N})$ such that $|\Phi(u)v| > \|u\|_\infty - \varepsilon$.**

(c) **What have we proved so far?**

(4) (a) **Prove that finitely supported sequences are dense in $\ell^p(\mathbb{N})$ for $p \geq 1$.**

(b) **Does the result hold in $\ell^\infty(\mathbb{N})$?**

(5) **For $n \in \mathbb{N}$, define the sequence e^n by $e^n_k = \delta_{k,n}$, that is $e^n = \{\overbrace{0, 0, \dots, 0}^{n}, 1, 0, 0, \dots\}$. Let $\varphi \in \ell^p(\mathbb{N})^*$ and $\gamma_n = \varphi(e^n)$. For $N \in \mathbb{N}$, define a sequence δ^N by**

$$\delta^N = \{\gamma_0|\gamma_0|^{q-2}, \gamma_1|\gamma_1|^{q-2}, \dots, \gamma_N|\gamma_N|^{q-2}, 0, 0, \dots\}.$$

(a) **Compute $\varphi(\delta^N)$.**

(b) **Prove that $\displaystyle\sum_{n=0}^{N} |\gamma_n|^q \leq \|\varphi\| \left(\sum_{n=0}^{N} |\gamma_n|^q\right)^{\frac{1}{p}}$.**

(c) **Deduce that the N-truncation of the sequence $\gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ has norm less than $\|\varphi\|$ in $\ell^q(\mathbb{N})$.**

(d) **Conclude that γ is in $\ell^q(\mathbb{N})$.**

(6) **Verify that $\varphi(u) = \Phi(\gamma)(u)$ if u is finitely supported and conclude.**

(7) **Prove the existence of a bounded linear functional on $\ell^\infty(\mathbb{N})$ that is not of the form $\Phi(u)$ with $u \in \ell^1(\mathbb{N})$.**
Hint: **consider the subspace C of convergent sequences and study the map $\lambda : v \mapsto \lim\limits_{n\to\infty} v_n$.**

Solution.
1. and 2. follow from the definitions and the Hölder Inequality.

(3) (a) Note that $\frac{1}{q-1} = \frac{p}{q}$. Direct computations show that $\|v\|_p = 1$ and that $\Phi(u)v = \|u\|_q$.

(b) Let $n_0 \in \mathbb{N}$ be such that $|u_{n_0}| > \|u\|_\infty - \varepsilon$ and define v by $v_n = \delta_{n,n_0}$.

(c) For each $u \in \ell^q(\mathbb{N})$, the linear functional $\Phi(u)$ is bounded on $\ell^p(\mathbb{N})$ and its operator norm is $\|u\|_q$. Therefore, Φ is an isometric embedding of $\ell^q(\mathbb{N})$ into $\ell^p(\mathbb{N})^*$ and all is left to prove is surjectivity.

(4) (a) If p is finite, every sequence in $\ell^p(\mathbb{N})$ is the limit of its truncations.

(b) No: non-zero constant sequences are not $\|\cdot\|_\infty$-limits of finitely supported sequences.

(5) (a) Observe that $\delta^N = \sum_{n=0}^N \gamma_n|\gamma_n|^{q-2}\, e^n$ so by linearity,

$$\varphi(\delta^N) = \sum_{n=0}^N \gamma_n|\gamma_n|^{q-2} \underbrace{\varphi(e^n)}_{=\gamma_n} = \sum_{n=0}^N |\gamma_n|^q.$$

(b) Since φ is continuous, the inequality $|\varphi(\delta^N)| \le \|\varphi\|\, \|\delta^N\|_p$ becomes

$$\sum_{n=0}^N |\gamma_n|^q \le \|\varphi\| \left(\sum_{n=0}^N |\gamma_n|^{(q-1)p}\right)^{\frac{1}{p}} = \|\varphi\| \left(\sum_{n=0}^N |\gamma_n|^q\right)^{\frac{1}{p}}.$$

(c) If γ^N denote the N-truncation of γ, the previous inequality reads

$$\|\gamma^N\|_q^q \le \|\varphi\|\, \|\gamma^N\|_q^{\frac{q}{p}}$$

that is, $\|\gamma^N\|_q^{q-\frac{q}{p}} \le \|\varphi\|$. Note that $q - \frac{q}{p} = 1$ to conclude.

(d) The bound on $\|\gamma^N\|_q$ is independent of the order of the truncation N so, letting $N \to \infty$, we see that $\|\gamma\|_q \le \|\varphi\| < \infty$.

(6) The identity $\varphi(u) = \Phi(\gamma)(u)$ for finitely supported u is a direct consequence of the definition of γ. Since φ and $\Phi(\gamma)$ are continuous and coincide on a dense subspace, they must be equal, so φ is in the range of Φ, which is therefore surjective. The argument (and the result!) fail for $p = \infty$ as we shall see in the next question.

(7) Observe that the linear functional λ is continuous on C with $\|\lambda\| = 1$. By Hahn-Banach, it extends to a bounded linear functional Λ on $\ell^\infty(\mathbb{N})$, with $\|\Lambda\| = 1$. Assume that Λ can be represented by $u \in \ell^1(\mathbb{N})$, that is $\Lambda(v) = \Phi(u)v$ for all $v \in \ell^\infty(\mathbb{N})$. Then, with the notation of question 5., we get $u_n = \Phi(u)e^n = \Lambda(e^n) = \lambda(e^n) = 0$ for all $n \ge 0$, so $u = 0$ hence $\lambda = 0$, which is excluded since $\|\lambda\| = 1$.

Problem set 5: Weak topologies

Problem 1.

Let E a Banach space, D a dense subset, $\{\varphi_n\}_{n\in\mathbb{N}}$ a sequence in E^* and $\varphi \in E^*$.

(1) **Prove that** $\varphi_n \xrightarrow[n\to\infty]{w^*} \varphi \Leftrightarrow \begin{cases} \{\varphi_n\}_{n\in\mathbb{N}} \text{ is bounded and} \\ \forall x \in D\,,\ \langle \varphi_n, x\rangle \underset{n\to\infty}{\longrightarrow} \langle \varphi, x\rangle \end{cases}$.

(2) **Can the boundedness assumption on $\{\varphi_n\}_{n\in\mathbb{N}}$ be removed?**

Solution.

(1) ($\Rightarrow$) The Uniform Boundedness principle implies that w^*-convergent sequences are bounded. ($\Leftarrow$) Since the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded, Alaoglu's Theorem implies the existence of a subsequence that converges to some $\psi \in E^*$, which, by the other assumption, coincides with φ on a dense subset.

(2) No. Consider for instance $E = c_0(\mathbb{N})$, with dense subset D the subspace of finitely supported sequences and $\varphi_n = n \cdot \mathrm{eval}_n$. Then φ_n converges to 0 pointwise on D but not on E.

Problem 2. For $n \geq 1$ and $a \leq x \leq b$, let $f_n(x) = \sin(nx)$.

(1) **Prove that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to 0 in $L^2([a,b])$.**
(2) **Does $\{f_n\}_{n\in\mathbb{N}}$ converge in $L^2([a,b])$?**

Hint: 1. All bounded linear forms on $L^2([a,b])$ are of the form $f \mapsto \int_a^b f(x)g(x)\,dx$ with $g \in L^2([a,b])$ and step functions are dense in $L^2([a,b])$.

Solution.

(1) This result goes by the name Riemann-Lebesgue Lemma. The equality
$$\lim_{n\to\infty} \int_a^b \sin(nx) f(x)\,dx = 0$$
is obvious if f is a step function and the result follows by density.

(2) If the sequence converged in $L^2([a,b])$ to some f, then it would converge weakly to the same limit, so f must be 0. However, the computation shows that
$$\|f_n\|_2^2 = \int_a^b \sin^2(nx)\,dx = \frac{b-a}{2} + \frac{\sin(2an) - \sin(2bn)}{4n} \underset{n\to\infty}{\not\to} 0.$$

Problem 3. Let $C_0(\mathbb{R}) = \{f \in C(\mathbb{R})\ ,\ \lim_{|x|\to\infty} f(x) = 0\}$.

(1) **Prove that $C_0(\mathbb{R})$ is closed in $L^\infty(\mathbb{R})$.**

(2) **Describe how $L^1(\mathbb{R})$ can be seen as a subspace of $C_0(\mathbb{R})^*$.**

(3) **Prove that every bounded sequence $\{u_n\}_{n\in\mathbb{N}}$ in $L^1(\mathbb{R})$ has a subsequence $\{u_{\varphi(n)}\}_{n\in\mathbb{N}}$ such that,**

$$\forall f \in C_0(\mathbb{R})\ ,\ \lim_{n\to\infty} \int_{\mathbb{R}} u_{\varphi(n)}(x) f(x)\, dx \textbf{ exists}.$$

(4) **Find the w^*-limit in $C_0(\mathbb{R})^*$ of the sequence $\{n\chi_n\}_{n\geq 1}$, where χ_n is the indicator of the interval $\left[-\frac{1}{n}, \frac{1}{n}\right]$.**

Solution.

(1) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $C_0(\mathbb{R})$ that converges to some f in L^∞. The sequence being Cauchy in L^∞ and the f_n's being continuous, the sequence is also Cauchy in every $(C([a,b]), \|\cdot\|_\infty)$, from which it follows that f can be assumed continuous, up to changing of representative, and that the convergence is uniform. To prove that f vanishes at infinity, fix $\varepsilon > 0$ and chose n large enough to have $\|f_n - f\|_\infty < \frac{\varepsilon}{2}$. Since $f_n \in C_0(\mathbb{R})$, there is a compact outside of which $|f_n(x)| < \frac{\varepsilon}{2}$ so that $|f(x)|$ cannot exceed ε outside of that same compact.

(2) The general idea is that taking dual spaces reverts inclusions:

$$\begin{array}{ccc} C_0(\mathbb{R}) & \subset & L^\infty(\mathbb{R}) \\ {}^*\downarrow & & \downarrow{}^* \\ C_0(\mathbb{R})^* & \supset & L^\infty(\mathbb{R})^* \underset{T}{\leftarrow} L^1(\mathbb{R}). \end{array}$$

Concretely, consider for $u \in L^1(\mathbb{R})$ the map $T_u(f) = \int_{\mathbb{R}} u(x)f(x)\,dx$ for $u \in C_0(\mathbb{R})$, then $|T_u(f)| \leq \|u\|_1 \|f\|_\infty$, so that $T_u \in C_0(\mathbb{R})^*$.

(3) With T_u defined as before for $u \in L^1(\mathbb{R})$, observe that $\|T_u\| \leq \|u\|_1$. So the sequence T_{u_n} is bounded and Alaoglu's Theorem implies that it admits a w^*-convergent subsequence, which is exactly the expected result.

(4) Let $f \in C_0(\mathbb{R})$. By continuity, for any $\varepsilon > 0$ there is an interval $[-\delta, \delta]$ on which $|f(x) - f(0)| \leq \varepsilon$. For $n > \delta^{-1}$, the average value of f on $\left[-\frac{1}{n}, \frac{1}{n}\right]$ (a.k.a $\frac{1}{2} T_{n\chi_n}(f)$) is within ε of $f(0)$. It follows that the w^*-limit of $\{T_{n\chi_n}\}_{n\in\mathbb{N}}$ is 2δ where δ denotes the Dirac measure at 0.

Problem set 6: C^*-algebras

Problem 1. Positivity in C^*-algebras

The purpose of this problem is to establish the following result:
Theorem. *Let $\mathcal{A}$ be a unital C^*-algebra. For $a \in \mathcal{A}$, the following statements are equivalent.*

(a) a is hermitian and $\mathrm{Sp}_{\mathcal{A}}(a) \subset [0, \infty)$
(b) There exists b in $\mathcal{A}$ such that $a = b^*b$
(c) There exists b hermitian in $\mathcal{A}$ such that $a = b^2$

An element satisfying **(a)** *is said* positive*and we write $a \geq 0$.*

(1) **What are the positive elements in $\mathbb{C}$? Verify that the theorem holds in this case.**

(2) **Let $a \in \mathcal{A}$ be hermitian. Prove that there exist positive elements u, v in $\mathcal{A}$ such that $a = u - v$ and $uv = vu = 0$.**

(3) **Let $a \geq 0$ in $\mathcal{A}$ and $n \in \mathbb{N}^*$. Prove the existence of $b \geq 0$ such that $a = b^n$.**

(4) **Verify that (a) $\Rightarrow$ (c) $\Rightarrow$ (b) in the theorem.**

(5) **We want to prove that the elements u, v and b in 2. and 3. are unique. Assume that $a = u' - v'$ with u', v' positive and $u'v' = v'u' = 0$.**

   (a) **Prove that $P(a) = P(u') + P(-v')$ for any polynomial P.**

   (b) **Let f be the function defined on $\mathbb{R}$ by $f(t) = \max(t, 0)$. Prove that $u = f(u') + f(-v')$.**

   (c) **Show that $f(u') = u'$ and $f(-v') = 0$ and conclude.**

   (d) **Use a similar method to prove that the element b in 3. is unique.**

Hints: 2.& 3. Functional Calculus, $t \mapsto \max(t, 0)$, $t \mapsto \max(-t, 0)$, $t \mapsto t^{\frac{1}{n}}$. 5.(a) Start with $P(t) = t^n$. For 5.(b), approach f uniformly on $\mathrm{Sp}(a) \cup \mathrm{Sp}(u') \cup \mathrm{Sp}(v')$ by polynomials.

Solution.

(1) Non-negative real numbers.

(2) Consider the continuous functions f and g defined on $\mathbb{R}$ by $f(t) = \max(t, 0)$ and $g(t) = \max(-t, 0)$. Since a is hermitian, $\mathrm{Sp}(a)$ is included in $\mathbb{R}$ so $f(a)$ and $g(a)$ are defined by the Functional Calculus and the Spectral Mapping Theorem implies that $u = f(a)$ and $v = g(a)$ are positive. Moreover, $f(t) - g(t) = t$ and $f(t)g(t) = 0$for all $t \in \mathbb{R}$ so $a = u - v$ and $uv = vu = 0$ since the Functional Calculus map is a morphism of algebras.

(3) Similarly, consider $h(t) = t^{\frac{1}{n}}$ on $\mathbb{R}^+$ and verify that $h(a)$ is a solution.

(4) **(a)** $\Rightarrow$ **(c)** follows from 3. with $n = 2$ and **(c)** $\Rightarrow$ **(b)** is tautological. A proof of the remaining implication can be found in Proposition 1.3.6 of Conway's book: *A Course in Abstract Analysis.*

(5)
 (a) The condition $u'v' = v'u' = 0$ implies that $a^n = u'^n + (-v')^n$ for $n \in \mathbb{N}$. The result follows by linear combination.

 (b) The subset $S = \mathrm{Sp}(a) \cup \mathrm{Sp}(u') \cup \mathrm{Sp}(v')$ is a compact of $\mathbb{R}$. By Stone-Weierstrass, there is a sequence $\{P_n\}_{n\in\mathbb{N}}$ of polynomials that converges uniformly to f on S. By continuity of the functional calculus map and the definition of u, the sequence $\{P_n(a)\}_{n\in\mathbb{N}}$ converges to u in $\mathcal{A}$. Since $P_n(a) = P_n(u') + P_n(-v')$, the result of the previous question implies that $u = f(u') + f(-v')$.

 (c) The relations $f(u') = u'$ and $f(-v') = 0$ directly follow from the definition of f. They imply that $u = u'$, which in turn implies that $v = v'$.

 (d) Assume that $a = b'^n$ and consider a sequence of polynomial functions Q_k converging to h uniformly on $\mathrm{Sp}(a) \cup \mathrm{Sp}(b')$. Passing to the limit in k in $Q_k(a) = Q_k(b'^n)$, we get $b = h(b'^n) = (h \circ k)(b')$ where $k(t) = t^n$ so that $h \circ k(t) = t$ on $\mathbb{R}^+ \supset \mathrm{Sp}(b')$ and $b = b'$.

Problem 2. Non-commutative topology.

If $X \xrightarrow{\varphi} Y$ is a continuous map between two topological spaces, we denote by $\varphi^\sharp$ the map from $C(Y)$ to $C(X)$ defined by

$$\varphi^\sharp(f) = f \circ \varphi.$$

(1) **Prove that $X \mapsto C(X)$, $\varphi \mapsto \varphi^\sharp$ is a contravariant functor from the category of compact Hausdorff spaces with continuous maps to the category of commutative unital C*-algebras with $*$-morphisms.**

(2) **What does the Gelfand-Naimark Theorem say about this functor? What more can be said?**

A map φ between locally compact Hausdorff spaces X and Y is said *proper* if the inverse image of a compact in Y is a compact of X.

(3) **Show that C_0 is a contravariant functor from the category of locally compact Hausdorff spaces to the category of commutative C*-algebras. Specify the morphisms.**

(4) **Prove that $C_0(X)$ is $*$-isomorphic to $C_0(Y)$ if and only if X and Y are homeomorphic.**

(5) **Assume X compact and $X_0 \subset X$ open.**

   (a) **Prove that $C_0(X_0)$ is an ideal of $C(X)$.**

   (b) **Show that all ideals in $C(X)$ are of this form.**

(6) **Complete the following 'dictionary' translating properties of topological spaces in terms of properties of algebras, commutative or not. You may restrict to the case of compact spaces whenever it makes sense.**

Spaces	**Algebras**
...	unital
points	...
...	ideals
...	quotients
...	$*$-morphism
...	$*$-isomorphism
disjoint union	...
connected component	...

Hints: maximal ideals in $C(X)$ are of the form $\mathcal{J}_{x_0} = \{f \in C(X)\,,\, f(x_0) = 0\}$. The spectrum of $C_0(X)$ is homeomorphic to X.
A *projection* in a C*-algebra is an element a that satisfies $a^2 = a^* = a$.

Solution.

(2) The functor $X \mapsto C(X)$ is essentially surjective by the Gelfand-Naimark Theorem and it is clearly faithful. It is also full: given a $*$-morphism $\Phi : C(Y) \longrightarrow C(X)$, consider the map between the maximal ideal spaces $\cdot \circ \Phi : \Sigma_{C(X)} \longrightarrow \Sigma_{C(Y)}$ and use the homeomorphism $\Sigma_{C(X)} \simeq X$. To sum up, $C(\cdot)$ is a contravariant equivalence of categories between compact Hausdorff spaces and commutative unital C*-algebras.

(3) The main point is that if f is in $C_0(Y)$ and φ is proper, then $f \circ \varphi$ is in $C_0(X)$. Consider, for $\varepsilon > 0$, a compact K_ε of Y outside of which $|f|$ does not exceed ε. Then, the same holds for $|f \circ \varphi|$ outside of $\varphi^{-1}(K_\varepsilon)$ which is compact by properness of φ.

(4) One direction follows from the fact that $\Sigma_{C_0(X)}$ is homeomorphic to X. For the other, let $X \xrightarrow{\varphi} Y$ be a homeomorphism and verify that $\varphi^\sharp$ is a $*$-isomorphism.

(5) (a) Let Y be the complement of X_0 in X. The kernel of the restriction morphism $f \mapsto f|_Y$ is exactly $C_0(X_0)$.

(b) Let $\mathcal{J}$ be an ideal in $C(X)$. Then $C(X)/\mathcal{J}$ is a commutative unital C*-algebra, hence of the form $C(Z)$ for some compact Hausdorff space Z by Gelfand-Naimark. Let π denote the natural projection $C(X) \longrightarrow C(Z)$. As in 2., there exists a map $\rho : Z \longrightarrow X$ such that $\rho^\sharp = \pi$. The surjectivity of π implies the injectivity of ρ and $\mathcal{J} = \ker \pi \simeq C_0(Y)$ where Y is the complement of $\rho(Z)$ in X.

(6) The last line in the table can be filled by remembering that a topological space X is connected if and only if any continuous function with values in $\{0,1\}$ is constant and observing that a projection in $C(X)$ is such a function.

Spaces	**Algebras**
compact	unital
points	maximal ideals
open subsets	ideals
closed subsets	quotients
proper map	$*$-morphism
homeomorphism	$*$-isomorphism
disjoint union	direct sum
connected component	projection

Problem set 7: (Pre-)Hilbert spaces

Problem 1. Gram-Schmidt orthonormalization

Let $\mathcal{X} = \{x_n\}_{n\geq 0}$ be a countable family of linearly independent vectors in a Hilbert space. Prove the existence of a countable orthonormal family $\mathcal{Y} = \{y_n\}_{n\geq 0}$ such that

$$\mathrm{Span}(x_0, \dots, x_p) = \mathrm{Span}(y_0, \dots, y_p)$$

for all $p \geq 0$.

Solution.
Let $y_0 = \frac{1}{\|x_0\|}x_0$. Assume constructed $y_0, \dots, y_n$ satisfying the requirements. The projection of x_{n+1} on $\mathrm{Span}(y_0, \dots, y_n)$ is $\sum_{k=0}^{n} \langle x_{n+1}, y_k\rangle\, y_k$ so

$$y'_{n+1} = x_{n+1} - \sum_{k=0}^{n} \langle x_{n+1}, y_k\rangle\, y_k$$

is orthogonal to all the vectors y_k for $k \leq n$. Switching y'_{n+1} and x_{n+1} accross the equality symbol and the induction hypothesis show the equality of the generated subspaces, and it suffices to define $y_{n+1} = \frac{1}{\|y'_{n+1}\|}y'_{n+1}$.

Problem 2. Orthogonal polynomials

Let I be an interval of $\mathbb{R}$ and $w : I \to \mathbb{R}$ a continuous positive function such that $x \mapsto x^n w(x)$ is integrable on I for any integer $n \geq 0$. Denote by $\mathcal{C}$ the set of continuous functions $f : I \to \mathbb{R}$ such that $x \mapsto f^2(x)w(x)$ is integrable. Finally, for f and g real-valued functions on I, we define

$$\langle f, g\rangle_w = \int_I f(x)g(x)w(x)\, dx.$$

(1) **Verify that $\mathbb{R}[X] \subset \mathcal{C}$ and that $\langle\cdot,\cdot\rangle_w$ is an inner product on $\mathcal{C}$. Denote by $\|\cdot\|_w$ the corresponding norm. Is $(\mathcal{C}, \|\cdot\|_w)$ a Hilbert space?**

(2) **Prove the existence of an orthonormal basis $\{P_n\}_{n\geq 0}$ of $\mathbb{R}[X]$ such that the degree of P_n is n and its leading coefficient γ_n is positive.**

(3) Verify that the polynomials P_n satisfy a relation of the form

$$P_n = (a_n X + b_n)P_{n-1} + c_n P_{n-2} \qquad (\dagger)$$

and determine the sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$.

(4) **Prove that P_n has n distinct roots in I.**

(5) **Assume I compact.**
  (a) **Find a constant C such that $\|f\|_w \leq C\|f\|_\infty$ for all $f \in \mathcal{C}$.**

  (b) **For f in $\mathcal{C}$, let $p_n(f)$ be the orthogonal projection of f on $\mathbb{R}_n[X]$. Prove that $p_n(f) \xrightarrow[n\to\infty]{\|\cdot\|_w} f$.**

Hint: 1. You may choose a concrete w to study completeness. 3. Project (†) and express a_n in terms of γ_n and γ_{n-1}. 4. Compute $\langle P_n, \prod_\alpha (X-\alpha)\rangle_w$ where the product is taken over roots of P_n with odd order.

Solution.

(1) Bilinearity comes from properties of the integral, positivity and definiteness result from the assumptions on the weight w. However, $(\mathcal{C}, \|\cdot\|_w)$ is not complete as discontinuous functions with finite $\|\cdot\|_w$ norm can be obtained as limits of Cauchy sequences in $\mathcal{C}$.

(2) Apply the Gram-Schmidt procedure to the canonical basis of $\mathbb{R}[X]$ and multiply by -1 if necessary to guarantee that $\gamma_n > 0$.

(3) First assume that such a relation exists. Then, projecting onto the lines generated by P_n, P_{n-1} and P_{n-2} leads to

$$1 = a_n\langle XP_{n-1}, P_n\rangle_w \quad 0 = a_n\langle XP_{n-1}, P_{n-1}\rangle_w + b_n \quad 0 = a_n\langle XP_{n-1}, P_{n-2}\rangle_w + c_n.$$

A direct computation shows that $a_n = \dfrac{\gamma_n}{\gamma_{n-1}}$. Similarly, $b_n = -\dfrac{\gamma_n}{\gamma_{n-1}}\langle XP_{n-1}, P_{n-1}\rangle$ and $c_n = -\dfrac{\gamma_{n-2}}{\gamma_{n-1}^2}$. Choosing these values for a_n, b_n and c_n guarantees that $P_n - ((a_nX + b_n)P_{n-1} + c_nP_{n-2})$ has degree at most $n-3$. This polynomial is a combination of P_n, P_{n-2} and XP_{n-1}. The first two are orthogonal to any P_k with $k \leq n-3$ by construction. For the last one, observe that $\langle XP_{n-1}, P_k\rangle = \langle P_{n-1}, XP_k\rangle = 0$ since $XP_k \in \operatorname{span}(P_0, \ldots, P_{n-2})$. This implies that $P_n - ((a_nX + b_n)P_{n-1} + c_nP_{n-2}) = 0$.

(4) Let $Q = \prod_\alpha (X-\alpha)$ where α runs over the roots odd order of P_n, with the convention that $Q = 1$ if there are no such roots. If Q has degree $< n$, then $P_n \perp Q$ by definition of the family $\{P_n\}$. On the other hand the function $x \mapsto P_n(x)Q(x)w(x)$ is non-negative so its integral is 0 only if it is constantly 0, which it is not. Therefore Q has degree n and P_n has n distinct roots in I.

(5) (a) A direct estimate gives $C = \sqrt{\int_I w}$. (b) Let $\varepsilon > 0$. By Stone-Weierstrass there exists a polynomial S such that $\|f - S\|_\infty < \frac{\varepsilon}{C}$. Let N be its degree. By optimality of the orthogonal projection, $\|f - p_N(f)\|_w \leq \|f - S\|_w \leq C\|f - S\|_\infty < \varepsilon$. Bessel's Inequality implies that $\{\|f - p_n(f)\|_w\}$ is a decreasing sequence and the result follows.

Note: families of orthogonal polynomials for various weights have many applications in a variety of contexts. In the case of $I = (-1, 1)$ with $w(x) = (1 - x^2)^{-\frac{1}{2}}$, one obtains the *Chebyshev polynomials of the first kind.* They are subject to the relation $P_n = 2xP_{n-1} - P_{n-2}$ and satisfy the relation $P_n(cos\theta) = \cos n\theta$. They are very useful in Approximation Theory. *Legendre polynomials* correspond to the case of $I = [-1, 1]$ with $w(x) = 1$, *Hermite polynomials* to the case of $I = \mathbb{R}$ with $w(x) = e^{-x^2}$ and *Laguerre polynomials* to the case of $I = [0, \infty)$ with $w(x) = e^{-x}$.

Problem 3. Let G be a group acting on a countable set X. Let $\mathcal{H} = \ell^2(X)$ be the Hilbert space of square-integrable functions on X for the counting measure.

(1) **Let A and B be subsets of X, with indicators denoted by χ_A and χ_B.**

  (a) **Give a condition on A, equivalent to $\chi_A \in \mathcal{H}$.**

  (b) **Give a condition on A and B, equivalent to $\chi_A \perp \chi_B$ in $\mathcal{H}$.**

(2) **For $f \in \mathcal{H}$ and $g \in G$, define $\pi(g)f = x \mapsto f(g^{-1} \cdot x)$.**

  (a) **Prove that each $\pi(g)$ is a unitary operator on $\mathcal{H}$.**

  (b) **Prove that $\pi : G \longrightarrow \mathrm{U}(\mathcal{H})$ is a group homomorphism.**

From now on, we assume that for every $x \in X$, the G-orbit $\{g \cdot x\ ,\ g \in G\}$ is infinite.

(3) **Let $A \subset X$ be such that $\chi_A \in \mathcal{H}$ and denote by C be the closure of the convex hull[1] of $C_0 = \{\pi(g)\chi_A\ ,\ g \in G\}$.**

  (a) **Prove the existence of a unique element ξ of minimal norm in C.**

  (b) **Verify that C is stable by each of the operators $\pi(g)$.**

  (c) **Prove that $\pi(g)\xi = \xi$ for all $g \in G$.**

  (d) **Deduce that ξ is constant on each G-orbit and conclude.**

[1] the convex hull of a set S is the family of all possible convex combinations of elements of S.

(4) **Let A, B be non-empty finite subsets of X and assume that $(g \cdot A) \cap B \neq \emptyset$ for all g in G.**

   (a) **Prove that $\langle f, \chi_B \rangle \geq 1$ for all $f \in C$.**

   (b) **Apply the previous result to ξ and conclude.**

Solution.

(1) Observe that $\langle \chi_A, \chi_B \rangle = \mathrm{Card}\, A \cap B$ so that $\chi_A \in \mathcal{H}$ if and only if A is finite and $\chi_A \perp \chi_B$ if and only if A and B are disjoint.

(2) Each map $x \mapsto g \cdot x$ is a bijection so the sums defining $\|f\|_2^2$ and $\|\pi(g)f\|_2^2$ only differ by the order of the terms. The morphism property follows from the fact that $(gh)^{-1} = h^{-1}g^{-1}$ in any group.

(3) (a) The set C is convex as the closure of a convex set and ξ is the projection of 0 on C.

   (b) By construction, C_0 is stable by each $\pi(g)$. These maps are continuous so C is stable too.

   (c) Since $\pi(g)$ is an isometry, $\|\pi(g)\xi\| = \|\xi\|$. Now ξ is the only element in C with norm $\|\xi\|$ so $\pi(g)\xi = \xi$.

   (d) We have $\xi(g^{-1} \cdot x) = \pi(g)\xi(x) = \xi(x)$ for all $x \in X$ and $g \in G$ so ξ is constant on the orbits. The only constant square-integrable function on an infinite discrete space is 0 so $\xi = 0$.

(4) The hypothesis on A and B implies that $\langle f, \chi_B \rangle \geq 1$ for all f of the form $\pi(g)\chi_A$. It extends to f in C_0 by convex combinations and to all of C by continuity of the inner product. In particular, we should have $\langle \xi, \chi_B \rangle \geq 1$, which contradicts the fact that $\xi = 0$.

Problem set 8: Fourier series

Problem 1. Pointwise and uniform convergence.

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a 2π-periodic function, piecewise continuous, piecewise of class C^1. For $x_0 \in \mathbb{R}$, we denote by $f(x_0^{\pm})$ the one-sided limit $\lim\limits_{x \to x_0^{\pm}} f(x)$ and $\tilde{f}$ is the function defined on $\mathbb{R}$ by

$$\tilde{f}(x) = \frac{f(x^+) + f(x^-)}{2}.$$

The purpose of the problem is to establish the pointwise convergence of the Fourier series of f to $\tilde{f}$, that is, for any $x_0 \in \mathbb{R}$,

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx_0} = \tilde{f}(x_0).$$

(1) **Verify that for any x_0 in $\mathbb{R}$, the map $h \mapsto \dfrac{f(x_0+h) + f(x_0-h) - f(x_0^+) - f(x_0^-)}{h}$ is bounded near 0.**

First, we consider the case $x_0 = 0$. Denote by $S_N(f)(0)$ the partial sum $\sum\limits_{n=-N}^{N} \hat{f}(n)$.

(2) **Prove that**

$$2\pi S_N(f)(0) = \int_0^{\pi} (f(x) + f(-x)) D_N(x)\, dx,$$

where $D_N(x)$ is the Dirichlet kernel $\dfrac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$.

(3) **Show that $2\pi(S_N(f)(0) - \tilde{f}(0))$ can be written as $\displaystyle\int_0^{\pi} g(x) \sin\left(N + \frac{1}{2}\right) x\, dx$ with g piecewise continuous and bounded near 0.**

(4) **Conclude and extend to the case of arbitrary x_0.**

From now on, we assume f continuous and piecewise of class C^1. We denote by φ the function defined on $\mathbb{R}$ by

$$\varphi(x) = \begin{cases} f'(x) & \text{if } f \text{ is differentiable at } x, \\[2ex] \dfrac{f'(x^+) + f'(x^-)}{2} & \text{otherwise.} \end{cases}$$

(5) **Verify the relation $\hat{\varphi}(n) = in\hat{f}(n)$ for all $n \in \mathbb{Z}$.**

(6) **Prove that the Fourier series of f converges normally to f.**

Hints: 4. Riemann-Lebesgue. Consider $f_{x_0} : x \mapsto f(x + x_0)$. 6. $|ab| \leq \frac{1}{2}(a^2 + b^2)$.

Solution.

(1) Boundedness follows from the existence of limits on the the left and the right for the function and its derivative.

(2) Partial sums of Fourier series are given by right convolution with Dirichlet's kernel, which is an even function.

(3) The function $g(x) = \dfrac{(f(x) + f(-x) - f(0^+) - f(0^-))}{\sin\left(\frac{x}{2}\right)}$ is bounded near 0 by the hypotheses and the fact that $\sin(x) \sim_0 x$.

(4) The integral converges to 0 as $N \to \infty$ by the Riemann-Lebesgue Lemma. For the general case, observe that $\widehat{f_{x_0}}(n) = e^{inx_0}\hat{f}(n)$.

(5) Integrate by parts on every interval where the function is of class C^1.

(6) For every n, we have $|\hat{f}(n)| = \left|\dfrac{\hat{\varphi}(n)}{n}\right| \leq \dfrac{1}{2}\left(|\hat{\varphi}(n)|^2 + \dfrac{1}{n^2}\right)$, summable by Parseval. Therefore, the series converges normally to its pointwise limit $\tilde{f}$.

Problem 2. Application to the computation of sums.

Let f be the 2π-periodic function on $\mathbb{R}$ defined by $f(x) = 1 - \dfrac{x^2}{\pi^2}$ for all $x \in [-\pi, \pi]$.

(1) **Compute the Fourier coefficients of f.**

(2) **Deduce the sums of the series $\displaystyle\sum_{n\geq 1} \frac{1}{n^2}$, $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{n^2}$ and $\displaystyle\sum_{n\geq 1} \frac{1}{n^4}$.**

Hints: note that only the real part of $\hat{f}(n)$ is useful. Parseval.

Solution.

A direct computation shows that $\hat{f}(0) = \dfrac{2}{3}$ and that the real part of $\hat{f}(n)$ is $\dfrac{2(-1)^{n+1}}{\pi^2 n^2}$. Since f clearly satisfies the hypotheses of the results proved in the previous problem, we get:

- $f(\pi) = 0 = \dfrac{2}{3} - \dfrac{2}{\pi^2} \displaystyle\sum_{|n|\geq 1} \dfrac{1}{n^2}$ so that $\displaystyle\sum_{n\geq 1} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.

- $f(0) = 1 = \dfrac{2}{3} - \dfrac{2}{\pi^2} \displaystyle\sum_{|n|\geq 1} \dfrac{(-1)^n}{n^2}$ so that $\displaystyle\sum_{n\geq 1} \dfrac{(-1)^n}{n^2} = -\dfrac{\pi^2}{12}$.

Finally, Parsevals' Identity gives $\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \left(1 - \dfrac{x^2}{\pi^2}\right)^2 dx = \dfrac{8}{15} = \dfrac{4}{9} + \dfrac{4}{\pi^4} \sum_{|n|\geq 1} \dfrac{1}{n^4}$ so that

$$\sum_{n\geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Problem 3. Not every function is equal to the sum of its Fourier series.

Let $\mathcal{C}_{2\pi}$ denote the space of 2π-periodic continuous functions on $\mathbb{R}$, equipped with $\|\cdot\|_\infty$. For $N \in \mathbb{N}$, we define a linear functional φ_N on $\mathcal{C}_{2\pi}$ by

$$\varphi_N(f) = S_N(f)(0) = \sum_{n=-N}^{N} \hat{f}(n).$$

(1) **Verify that $\mathcal{C}_{2\pi}$ is a Banach space.**

(2) **Prove that $\varphi_N \in \mathcal{C}_{2\pi}^*$ and compute $\|\varphi_N\|$.**

(3) **Show that $\|\varphi_N\| \geq \dfrac{2}{\pi} \displaystyle\int_0^{\frac{(2N+1)\pi}{2}} \left|\dfrac{\sin u}{u}\right| du$ for any $N \in \mathbb{N}$.**

(4) **Prove the existence of a function in $\mathcal{C}_{2\pi}$ whose Fourier series diverges at 0.**

Hints: 2. Consider $f_\varepsilon = \frac{D_N}{|D_N|+\varepsilon}$. 4. Use the Principle of Uniform Boundedness.

Solution.

(1) The space $\mathcal{C}_{2\pi}$ is a closed subspace of the Banach space of bounded functions on $\mathbb{R}$.

(2) Using the Dirichlet kernel once more, we see that $\varphi_N(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_N(x)\,dx$ from which it follows that $\|\varphi_N\| \leq \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |D_N(x)|\,dx$, so that $\varphi_N \in \mathcal{C}_{2\pi}^*$. To prove the reverse inequality, consider $f_\varepsilon = \frac{D_N}{|D_N|+\varepsilon}$ for $\varepsilon > 0$. It is clearly in the unit ball and $\displaystyle\lim_{\varepsilon\to 0}\varphi_N(f_\varepsilon) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)|\,dx$ so finally,

$$\|\varphi_N\| = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)|\,dx.$$

(3) It follows from the inequality $\left|\sin\left(\dfrac{x}{2}\right)\right| \leq \left|\dfrac{x}{2}\right|$ and the change of variables $u = \left(N + \dfrac{1}{2}\right)x$

(4) The improper integral $\displaystyle\int_0^\infty \left|\frac{\sin u}{u}\right| du$ is divergent so $\displaystyle\lim_{N\to\infty}\|\varphi_N\| = \infty$. If $\varphi_N(f)$ was convergent for all $f \in \mathcal{C}_{2\pi}$, the Principle of Uniform Boundedness would imply that $\|\varphi_N\|$ is a bounded sequence, so there exist functions whose Fourier series must diverge at 0.
Note that such functions can be expliciteely constructed, see for instance Chapter 3 in [SS].

In-class midterm

Solution p.42

This exam consists of three independent problems. You may treat them in the order of your choosing.

If you were not able to solve a question but wish to use the result to solve another one, you are welcome to do so, as long as you indicate it explicitly.

Notation: if (E, d) is a metric space, $x \in E$ and $r > 0$, we denote by $B_E(x, r)$ the *open* ball centered at x with radius r, that is,

$$B_E(x, r) = \{y \in E \ , \ d(x, y) < r\}.$$

Reminder: a useful consequence of the Baire Category Theorem is the following.

Proposition. *If E is a Baire space and $\{F_n\}_{n\geq 1}$ is a sequence of closed subsets such that $\bigcup_{n\geq 1} F_n = E$, then $\bigcup_{n\geq 1} \overset{\circ}{F_n}$ is a dense open subset of E.*

Problem 1

1. Is $c_0(\mathbb{N}) = \Big\{\{u_n\} \in \mathbb{R}^{\mathbb{N}} \ , \ \lim_{n\to\infty} u_n = 0\Big\}$ complete for the norm $\|\{u_n\}\|_\infty = \sup_{n\in\mathbb{N}} |u_n|$?

2. Is $C([0, 1], \mathbb{R})$ complete for the norm $\|f\|_1 = \displaystyle\int_0^1 |f(x)|\, dx$?

Problem 2

Let E and F be Banach spaces. We denote by $\mathbb{B}$ the closed ball of radius 1 in E, that is, $\mathbb{B} = \overline{B_E(0,1)}$. A bounded operator $T \in \mathcal{L}(E,F)$ is said *compact* if $\overline{T(\mathbb{B})}$ is compact.

1. Characterize the Banach spaces E such that the identity map Id_E is compact.

2. Assume that $T \in \mathcal{L}(E,F)$ has finite-dimensional range. Prove that T is compact.

3. Let $T \in \mathcal{L}(E,F)$ be compact and assume that the range $r(T)$ of T is closed in F.

a. Show the existence of $\rho > 0$ such that $B_{r(T)}(0,\rho) \subset T(\mathbb{B})$.

b. Prove that $r(T)$ is finite-dimensional.

4. *Integral operators with continuous kernels are compact.*
Let $E = (C([0,1]), \|\cdot\|_\infty)$. For $\kappa \in C([0,1]^2)$, we define a linear map $T : E \longrightarrow E$ by

$$T(f)(x) = \int_0^1 \kappa(x,y) f(y)\, dy.$$

a. Prove that T is continuous.

b. Prove that T is compact.

Problem 3

1. Let (E,d) and (F,δ) be metric spaces. Assume E complete and consider a sequence $\{f_n\}_{n\geq 1}$ of continuous maps from E to F that converges pointwise to $f : E \longrightarrow F$.

a. Consider, for $n \geq 1$ and $\varepsilon > 0$, the set

$$F_{n,\varepsilon} = \{x \in E \quad \text{s. t.} \quad \forall p \geq n\ ,\ \delta(f_n(x), f_p(x)) \leq \varepsilon\}\,.$$

Show that $\Omega_\varepsilon = \bigcup_{n\geq 1} \overset{\circ}{F}_{n,\varepsilon}$ is a dense open subset of E.

b. Show that every point $x_0 \in \Omega_\varepsilon$ has a neighborhood $\mathcal{N}$ such that

$$\forall x \in \mathcal{N}\ ,\ \delta(f(x_0), f(x)) \leq 3\varepsilon.$$

c. Prove that f is continuous at every point of $\Omega = \bigcap_{n\geq 1} \Omega_{\frac{1}{n}}$ and that $\overline{\Omega} = E$.

2. *Application:* let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Show that its derivative f' is continuous on a dense subset of $\mathbb{R}$.

Math 113 - In-class midterm

Solution

Notation: if (E,d) is a metric space, $x \in E$ and $r > 0$, we denote by $B_E(x,r)$ the *open* ball centered at x with radius r, that is, $B_E(x,r) = \{y \in E \ , \ d(x,y) < r\}$.

Reminder: a useful consequence of the Baire Category Theorem is the following.

Proposition. *If E is a Baire space and $\{F_n\}_{n\geq 1}$ is a sequence of closed subsets such that $\bigcup_{n\geq 1} F_n = E$, then $\bigcup_{n\geq 1} \overset{\circ}{F_n}$ is a dense open subset of E.*

Problem 1

1. Is $c_0(\mathbb{N}) = \big\{\{u_n\} \in \mathbb{R}^{\mathbb{N}} \ , \ \lim_{n\to\infty} u_n = 0\big\}$ complete for the norm $\|\cdot\|_\infty$?

Yes. Note that it is enough to prove that $c_0(\mathbb{N})$ is closed in $\ell^\infty(\mathbb{N})$, which is complete for the given norm. One may also proceed directly: let $\{u^p\}_{p\in\mathbb{N}}$ be a Cauchy sequence in $c_0(\mathbb{N})$. For $\varepsilon > 0$, there exists a rank N_ε such that $\|u^p - u^q\|_\infty < \frac{\varepsilon}{2}$ for $p, q \geq N_\varepsilon$, that is,

$$(\dagger) \qquad \forall n \in \mathbb{N} \ , \ |u_n^p - u_n^q| < \frac{\varepsilon}{2}.$$

This means that given n fixed, the sequence $\{u_n^p\}_{p\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ complete. Denote $u_n = \lim_{p\to\infty} u_n^p$. We shall prove that

(1) the sequence u belongs to $c_0(\mathbb{N})$,
(2) the convergence occurs for the norm $\|\cdot\|$.

(1) To see that u vanishes at infinity, observe that the Triangle Inequality gives

$$|u_n| \leq |u_n^p| + |u_n - u_n^p|.$$

Fix $p > N_\varepsilon$ and let $q \to \infty$ in ($\dagger$) to get $|u_n - u_n^p| \leq \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$. Since $u^p \in c_0(\mathbb{N})$, there exists N'_ε such that $n > N'_\varepsilon$ implies $|u_n^p| < \frac{\varepsilon}{2}$ which guarantees $|u_n| < \varepsilon$.
(2) As before, fix $p > N_\varepsilon$, let $q \to \infty$ in ($\dagger$) and note that N_ε does not depend on n to see that the convergence is uniform.

2. Is $C([0,1],\mathbb{R})$ complete for the norm $\|f\|_1 = \displaystyle\int_0^1 |f(x)|\,dx$?

No. Consider for instance (= draw a picture of) the sequence of continuous functions f_n where

$$f_n(x) = \begin{cases} 0 & \text{if} \quad x \leq \frac{1}{2} \\ 1 & \text{if} \quad x \geq \frac{1}{2} + \frac{1}{n} \end{cases}$$

and f_n is affine on $(\frac{1}{2}, \frac{1}{2} + \frac{1}{n})$. Check that $\{f_n\}_{n\in\mathbb{N}}$

- is Cauchy with respect to $\|\cdot\|_1$;
- converges pointwise to the discontinuous function f that is constantly 0 on $[0, \frac{1}{2}]$ and constantly 1 on $(\frac{1}{2}, 1]$.

Prove that $\lim\limits_{n\to\infty} \|f_n - f\|_1 = 0$ to conclude.

Problem 2

Let E and F be Banach spaces. We denote by $\mathbb{B}$ the closed ball of radius 1 in E, that is, $\mathbb{B} = \overline{B_E(0,1)}$. A bounded operator $T \in \mathcal{L}(E,F)$ is said *compact* if $\overline{T(\mathbb{B})}$ is compact. The range of T is denoted by $r(T)$.

1. Characterize the Banach spaces E such that the identity map Id_E is compact.

Riesz's Theorem asserts that Id_E is compact if and only if E is finite-dimensional.

2. Let $T \in \mathcal{L}(E,F)$ with $r(T)$ finite-dimensional. Prove that T is compact.

By the assumption on $r(T)$, it suffices to prove that $\overline{T(\mathbb{B})}$ is closed and bounded. Closedness holds by definition. Boundedness follows from the continuity of T: by definition of the operator norm, $T(\mathbb{B}) \subset B_{r(T)}(0, \|T\|)$ so $\overline{T(\mathbb{B})} \subset \overline{B_{r(T)}(0, \|T\|)}$.

3. Let $T \in \mathcal{L}(E,F)$ be compact and assume that $r(T)$ of T is closed in F.

a. Show the existence of $\rho > 0$ such that $B_{r(T)}(0,\rho) \subset T(\mathbb{B})$.

The operator T induces a surjective continuous linear map $\tilde{T} : E \longrightarrow r(T)$. Since $r(T)$ is closed in F Banach, it is complete so the Open Mapping Theorem applies. Consider for instance the open ball $B_E(0,1)$. Since, $\tilde{T}$ is open, $\tilde{T}(B_E(0,1))$ is an open subset of $r(T)$ that contains 0_F so it must contain a ball centered at 0_F, say

$$B_{r(T)}(0,\rho) \subset \tilde{T}(B_E(0,1)) \subset T(\mathbb{B}).$$

b. Prove that $r(T)$ is finite-dimensional.

Taking closures in the previous inclusion, the closed ball $\overline{B_{r(T)}(0,\rho)}$ is closed in $\overline{T(\mathbb{B})}$, compact by assumption, hence compact itself. Since the dilation by ρ^{-1} is continuous, it follows that $\overline{B_{r(T)}(0,1)}$ is compact, so that Riesz's Theorem implies that $r(T)$ is finite-dimensional.

4. Let $E = (C([0,1]), \|\cdot\|_\infty)$. For $\kappa \in C([0,1]^2)$, we define a linear map $T : E \longrightarrow E$ by

$$T(f)(x) = \int_0^1 \kappa(x,y) f(y)\, dy.$$

a. Prove that T is continuous.

The kernel κ is continuous on the compact $[0,1]^2$ so it is bounded and one can verify that $\|\kappa\|_\infty$ is a Lipschitz constant for T.

b. Prove that T is compact.

The same arguments as in **2.** show that $\overline{T(\mathbb{B})}$ is closed and bounded. By Arzelà-Ascoli, it suffices to prove that $T(\mathbb{B})$ is equicontinuous. This follows from the uniform continuity of κ on the compact $[0,1]^2$: for $0 \le x, z \le 1$ and $f \in \mathbb{B}$,

$$|T(f)(x) - T(f)(z)| \le \|f\|_\infty \int_0^1 |\kappa(x,y) - \kappa(z,y)|\, dy.$$

Since κ is uniformly continuous, there exists $\delta > 0$ such that $|x - z| < \delta$ implies that $|\kappa(x, y) - \kappa(z, y)| < \varepsilon$ for all x, y, z such that $|x - z| < \delta$. For such x and z, we get $|T(f)(x) - T(f)(z)| \leq \varepsilon$, so the family $\{T(f)\ ,\ f \in \mathbb{B}\}$ is equicontinuous.

Problem 2

1. Let (E, d) and (F, δ) be metric spaces. Assume E complete and consider a sequence $\{f_n\}_{n\geq 1}$ of continuous maps from E to F that converges pointwise to $f : E \longrightarrow F$.

a. Consider, for $n \geq 1$ and $\varepsilon > 0$, the set $F_{n,\varepsilon} = \{x \in E\ , \forall p \geq n\ ,\ \delta(f_n(x), f_p(x)) \leq \varepsilon\}$. Show that $\Omega_\varepsilon = \bigcup_{n\geq 1} \overset{\circ}{F}_{n,\varepsilon}$ is a dense open subset of E.

According to the consequence of the Baire Category Theorem recalled above, it suffices to prove that the sets $F_{n,\varepsilon}$ are closed and cover E. For given n and p, the set $\{x \in E\ ,\ \delta(f_n(x), f_p(x)) \leq \varepsilon\}$ is closed as the inverse image of $[0, \varepsilon]$, closed, under the map $x \mapsto \delta(f_n(x), f_p(x))$, continuous as composed of continuous functions. Taking the intersection over $p \geq n$ gives $F_{n,\varepsilon}$ closed. That the union of these sets covers E follows from the pointwise convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$.

b. Show that every point $x_0 \in \Omega_\varepsilon$ has a neighborhood $\mathcal{N}$ such that

$$\forall x \in \mathcal{N}\ ,\ \delta(f(x_0), f(x)) \leq 3\varepsilon.$$

Let n be such that $x_0 \in \overset{\circ}{F}_{n,\varepsilon}$. Since $\overset{\circ}{F}_{n,\varepsilon}$ is open and f_n is continuous, there exists a neighborhood $\mathcal{N}$ of x_0 included in $\overset{\circ}{F}_{n,\varepsilon}$ such that

$$\delta(f_n(x_0), f_n(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N}.$$

Since $\mathcal{N} \subset \overset{\circ}{F}_{n,\varepsilon}$, we have

$$\delta(f_n(x), f_p(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N} \text{ and } p \geq n.$$

Letting $p \to \infty$ in this inequality,

$$\delta(f_n(x), f(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N}.$$

Now, by the triangle inequality we get, for all $x \in \mathcal{N}$,

$$\begin{aligned} \delta(f(x), f(x_0)) &\leq \delta(f(x), f_n(x)) + \delta(f_n(x), f_n(x_0)) + \delta(f_n(x_0), f(x_0)) \\ &\leq \varepsilon + \varepsilon + \varepsilon. \end{aligned}$$

c. Prove that f is continuous at every point of $\Omega = \bigcap_{n\geq 1} \Omega_{\frac{1}{n}}$ and that $\overline{\Omega} = E$.

Let $x_0 \in \Omega$ and $\varepsilon > 0$. Fix n such that $\frac{1}{n} < \frac{\varepsilon}{3}$. By the previous result, there is a neighborhood $\mathcal{N}$ of x_0 such that $\delta(f(x), f(x_0)) \leq \varepsilon$ for all $x \in \mathcal{N}$, which proves continuity of f at x_0. The fact that Ω is dense in E follows from **a.** and the Baire Category Theorem.

2. Let f be differentiable on $\mathbb{R}$. Show that f' is continuous on a dense set.

Apply the previous result to the sequence $f_n : x \mapsto \frac{f\left(x+\frac{1}{n}\right)-f(x)}{1/n}$.

Take-home midterm

Solution p.46

The goal of this problem is to give a proof of the following density result.

Theorem. (Weierstrass) *Every continuous function on a segment of the real line is the uniform limit of a sequence of polynomial functions.*

0. The theorem asserts in particular that the family of functions $\{x \mapsto x^n\}_{n\in\mathbb{N}}$ is a topological basis of $(C([0,1]), \|\cdot\|_\infty)$. Is it an algebraic basis?

Let $\mathcal{E}$ be the space of continuous and compactly supported complex-valued functions on $\mathbb{R}$. For $f, g \in \mathcal{E}$, let $f \star g$ denote the *convolution product of f and g*, defined by

$$f \star g\,(x) = \int_{\mathbb{R}} f(t)g(x-t)\,dt.$$

1. Verify that $(\mathcal{E}, +, \star)$ is an algebra. Is it unital?

Definition. An *approximate unit* in $\mathcal{E}$ is a sequence $\{\chi_n\}_{n\geq 1}$ such that for any f in $\mathcal{E}$, the sequence $\{\chi_n \star f\}$ converges uniformly to f.

2. Sketch the graphs of functions α_n in $\mathcal{E}$ such that

- $\forall n \geq 1$, α_n only takes non-negative values;
- $\forall n \geq 1$, $\displaystyle\int_{\mathbb{R}} \alpha_n(t)\,dt = 1$;
- $\forall A > 0$, $\displaystyle\lim_{n\to\infty} \int_{|t|\geq A} \alpha_n(t)\,dt = 0$;

and prove that the sequence $\{\alpha_n\}_{n\geq 1}$ is an approximate unit in $\mathcal{E}$.

3. Define, for $n \geq 1$, $a_n = \displaystyle\int_{-1}^{1} (1-t^2)^n\,dt$ and $p_n : t \longmapsto \begin{cases} \dfrac{(1-t^2)^n}{a_n} & \text{if } |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

Show that $\{p_n\}_{n\geq 1}$ is an approximate unit in $\mathcal{E}$.

4. Let f be a function in $\mathcal{E}$ that vanishes outside of $[-\frac{1}{2}, \frac{1}{2}]$. Prove that, for every $n \geq 1$, the function $p_n \star f$ is polynomial on its support.

5. Prove Weierstrass' Theorem.

Math 113 - Take-home midterm

Solution

The goal of the problem is to give a proof of the following density result.

Theorem. (Weierstrass) *Every continuous function on a segment of the real line is the uniform limit of a sequence of polynomial functions.*

0. The theorem asserts in particular that the family of functions $\{x \mapsto x^n\}_{n\in\mathbb{N}}$ is a topological basis of $(C([0,1]), \|\cdot\|_\infty)$. Is it an algebraic basis?

No: linear combinations of monomials are smooth while some continuous functions fail to be differentiable.

Let $\mathcal{E}$ be the space of continuous and compactly supported complex-valued functions on $\mathbb{R}$. For $f, g \in \mathcal{E}$, let $f \star g$ denote the *convolution product of f and g*, defined by

$$f \star g\,(x) = \int_{\mathbb{R}} f(t)g(x-t)\,dt.$$

1. Verify that $(\mathcal{E}, +, \star)$ is an algebra. Is it unital?

The verification is routine, using Fubini and changes of variables. Note that $\mathrm{supp}(f * g) \subset \mathrm{supp}(f) + \mathrm{supp}(g)$. Assume that $(\mathcal{E}, +, \star)$ is unital. Then, there exists a continuous function f such that $f * g = g$ for all $g \in \mathcal{E}$. In particular, the relation $f * g(0) = g(0)$ implies that $\int_{\mathbb{R}} f(t)h(t)\,dt = h(0)$ for any $h \in \mathcal{E}$. Since f cannot be identically zero, assume that it takes a positive value at $x_0 \neq 0$. Then there exists $\delta > 0$ such that $0 < x_0 - \delta$ and f only takes positive values on $I = [x_0 - \delta, x_0 + \delta]$. Consider h supported in I, non-negative and not identically zero. Then $h(0) = 0 \neq \int_{\mathbb{R}} f(t)h(t)\,dt$, which contradicts the assumption on f. Therefore f must vanish everywhere except perhaps at 0, but since it must be continuous, it is constantly zero.

Definition. An *approximate unit* in $\mathcal{E}$ is a sequence $\{\chi_n\}_{n\geq 1}$ such that for any f in $\mathcal{E}$, the sequence $\{\chi_n \star f\}$ converges uniformly to f.

2. Prove that a sequence of non-negative functions α_n in $\mathcal{E}$ such that

$$\forall n \geq 1 \quad , \quad \int_{\mathbb{R}} \alpha_n(t)\,dt = 1 \qquad \textbf{and} \qquad \forall A > 0 \quad , \quad \lim_{n\to\infty} \int_{|t|\geq A} \alpha_n(t)\,dt = 0$$

is an approximate unit.

Let $f \in \mathcal{E}$. Since f is continuous and compactly supported, it is uniformly continuous. Fix $\varepsilon > 0$ and let $\eta > 0$ be such that

$$|x-y| < \eta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Then, if n is large enough so that $\displaystyle\int_{|t|\geq\eta} \alpha_n(t)\,dt < \varepsilon$,

$$\begin{aligned}
|f * \alpha_n(x) - f(x)| &= \left|\int_{\mathbb{R}} \left(f(x-t) - f(x)\right)\alpha_n(t)\right|\,dt \\
&\leq \int_{|t|>\eta} |f(x-t) - f(x)|\alpha_n(t)\,dt + \int_{-\eta}^{\eta} |f(x-t) - f(x)|\alpha_n(t)\,dt \\
&< 2\|f\|_\infty\varepsilon + \varepsilon\int_R \alpha_n(t)\,dt = (2\|f\|_\infty + 1)\varepsilon,
\end{aligned}$$

which can be made arbitrarily small, independently of x.

3. Define, for $n \geq 1$, $a_n = \displaystyle\int_{-1}^{1} (1-t^2)^n\,dt$ and $p_n : t \longmapsto \begin{cases} \dfrac{(1-t^2)^n}{a_n} & \textbf{if } |t| \leq 1 \\ 0 & \textbf{otherwise} \end{cases}$.

Show that $\{p_n\}_{n\geq 1}$ is an approximate unit in $\mathcal{E}$.

The non-negativity and normalization are immediate. Note that $\displaystyle\int_{|t|\geq A} p_n(t)\,dt = 0$ if $A \geq 1$ and that

$$a_n = 2\int_0^1 (1-t^2)^n\,dt \geq 2\int_0^1 (1-t)^n\,dt = \frac{2}{n+1}.$$

For $0 < A < 1$ and $n \geq 1$, we see that

$$\begin{aligned}
\int_{|t|\geq A} p_n(t)\,dt &= \frac{2}{a_n}\int_A^1 (1-t^2)^n\,dt \\
&\leq \frac{2}{a_n}(1-A^2)^n \\
&= (n+1)\underbrace{(1-A^2)}_{<1}{}^n \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

so $\{p_n\}_{n\geq 1}$ is an approximate unit in $\mathcal{E}$.

4. Let f be a function in $\mathcal{E}$ that vanishes outside of $[-\frac{1}{2}, \frac{1}{2}]$. Prove that, for every $n \geq 1$, the function $p_n \star f$ is polynomial on its support.

First observe that $p_n(x - t)$ is a polynomial in x. To fix notations, we write

$$p_n(x-t) = \sum_{k=0}^{2n} c_k(t) x^k.$$

Then, for x in the support of the convolution,

$$(f * p_n)(x) = \sum_{k=0}^{2n} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) c_k(t)\, dt \right) x^k$$

which is a polynomial expression.

5. Prove Weierstrass' Theorem.

It follows from the previous results that a continuous function with compact support in $[-\frac{1}{2}, \frac{1}{2}]$ is a uniform limit of polynomial functions. Now let f be a continuous function defined on a segment $[a, b]$. Extend f to a function $\tilde{f} \in \mathcal{E}$. This can be done for instance by requesting that $\tilde{f}$ be 0 oustide of $[a-1, b+1]$, coincide with f on $[a, b]$ and affine elsewhere.
An affine transformation from $[a-1, b+1]$ to $[-\frac{1}{2}, \frac{1}{2}]$ allows to use the result proved in **4.** and to conclude.

Final examination

Duration: 4 hours

Problem 1

Let $\{\alpha_n\}_{n\geq 0}$ be a bounded sequence of complex numbers and S defined on $\ell^2(\mathbb{N})$ by

$$S(u_0, u_1, \ldots) = (0, \alpha_0 u_0, \alpha_1 u_1, \ldots).$$

1. Verify that $S \in \mathcal{B}(\ell^2(\mathbb{N}))$ and compute its operator norm.

2. Determine the adjoint of S.

3. Is S a normal operator?

4. Are there sequences $\{\alpha_n\}_{n\geq 0}$ such that S is an isometric embedding? An isometry?

Problem 2

Let $\mathcal{H}$ be a separable Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. Define an operator $A^\dagger$ on $\mathcal{H}^*$ by

$$A^\dagger \varphi = \varphi \circ A.$$

1. Verify that $A^\dagger \in \mathcal{B}(\mathcal{H}^*)$.

2. What is the relation between $A^\dagger$ and the adjoint A^* of A?

Problem 3

Let $\mathcal{H}$ be a separable Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Recall that if T is **hermitian**, then

$$\|A\| = \sup\{|\langle T\xi \,|\, \xi\rangle|\,,\ \|\xi\| = 1\}.$$

1. Assume T hermitian such that $T\xi \perp \xi$ for all $\xi \in \mathcal{H}$.

a. What can be said of T?

b. Does the result hold if T is not hermitian?

Let A be an arbitrary element of $\mathcal{B}(\mathcal{H})$

2. Find an operator $B \in \mathcal{B}(\mathcal{H})$ such that $\|A\xi\|^2 - \|A^*\xi\|^2 = \langle B\xi \,|\, \xi\rangle$ for all $\xi \in \mathcal{H}$.

3. Prove that A is normal if and only if $\|A\xi\| = \|A^*\xi\|$ for all $\xi \in \mathcal{H}$.

Problem 4

Let $\mathcal{A}$ be a unital Banach algebra, with unit denoted by 1, and a, b elements of $\mathcal{A}$.

1. Let $\lambda \in \mathbb{C}^\times$ be such that $\lambda - ab$ is invertible. Prove that $\lambda - ba$ is invertible, with inverse

$$\lambda^{-1} + \lambda^{-1} b(\lambda - ab)^{-1} a.$$

2. Prove that $\mathrm{Sp}(ab) \cup \{0\} = \mathrm{Sp}(ba) \cup \{0\}$.

3. Prove that ab and ba have the same spectral radius.

4. Give an example of elements a and b in a Banach algebra such that $\mathrm{Sp}(ab) \neq \mathrm{Sp}(ba)$.

Problem 5

Let E and F be Banach spaces.

1. Prove that every weakly convergent sequence in E is bounded.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a weakly convergent sequence in E, with weak limit x and $T : E \longrightarrow F$ a bounded linear map.

2. Prove that the sequence $\{Tx_n\}_{n\in\mathbb{N}}$ converges weakly to Tx.

From now on, assume that T is **compact**, that is, the image by T of the closed unit ball of E is compact in F.

3. Show that every subsequence of $\{Tx_n\}_{n\in\mathbb{N}}$ has a subsequence that converges (strongly) to Tx.

4. Conclude that Tx_n converges strongly to Tx.