MATH 81/111: RINGS AND FIELDS
HOMEWORK #8

This homework set is required for students enrolled in Math 111 and is optional (strongly encouraged) for those in Math 81.

Problem 8.1. Let Ω/F be an extension of fields and let $G = \operatorname{Aut}_F(\Omega)$ be equipped with the Krull topology, with the basis of open sets

$$V(\sigma, S) = \{\tau \in G : \tau|_S = \sigma|_S\}$$

for $\sigma \in G$ and $S \subseteq \Omega$ finite.

(a) Show that the inverse map $G \to G$ by $\sigma \mapsto \sigma^{-1}$ is continuous.

(b) Let $X \subseteq G$ be a subset. Show that the closure $\operatorname{cl}(X)$ of X in G is equal to

$$\operatorname{cl}(X) = \{\sigma \in G : \text{for all finite } S \subseteq \Omega, \text{ there exists } \tau \in X \text{ such that } \tau|_S = \sigma|_S\}.$$

(c) Let $H \leq G$ be a subgroup (not necessarily closed). Show that H and $\operatorname{cl}(H)$ have the same fixed subfield, i.e.,

$$\Omega^H = \Omega^{\operatorname{cl}(H)} \subseteq \Omega.$$

Problem 8.2. Consider the direct product of groups

$$\prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}$$

equipped with the product topology and each $\mathbb{Z}/n\mathbb{Z}$ the discrete topology. Let

$$\widehat{\mathbb{Z}} = \{(a_n)_{n \geq 1} : a_n \equiv a_m \pmod{m} \text{ if } m \mid n\} \subseteq \prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}.$$

(a) Show that $\widehat{\mathbb{Z}}$ (with the subspace topology) is a compact subgroup of $\prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}$.

(b) Show that the map $\mathbb{Z} \to \widehat{\mathbb{Z}}$ by $1 \mapsto (1, 1, \dots)$ is a group homomorphism with dense image, giving $\mathbb{Z}$ the subspace topology.

(c) Using the map

$$\operatorname{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \hookrightarrow \prod_{n=1}^{\infty} \operatorname{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$$

show that there is an isomorphism of topological groups

$$\operatorname{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \xrightarrow{\sim} \widehat{\mathbb{Z}}$$

i.e., a group homomorphism that is also a homeomorphism of topological spaces. *[Hint: See also Example 7.16 for some inspiration.]*

Problem 8.3. Let Ω/F be a Galois extension of fields (finite or infinite). Show that $\operatorname{Gal}(L/K)$ is either finite or uncountable.

Date: 25 February 2015; due Monday, 9 March 2015.