MATH 81/111: RINGS AND FIELDS
HOMEWORK #4

Problem 4.1. Consider the polynomial $f(X) = X^4 - 2X^2 - 2$. By the quadratic formula, the roots of f are

$$\pm\sqrt{1 \pm \sqrt{3}}.$$

(a) Let $K_1 = \mathbb{Q}(\sqrt{1+\sqrt{3}})$ and $K_2 = \mathbb{Q}(\sqrt{1-\sqrt{3}})$. Show that $K_1 \neq K_2$ (as subfields of $\mathbb{C}$) and $K_1 \cap K_2 = \mathbb{Q}(\sqrt{3})$.
(b) Let $F = K_1 \cap K_2$. Show that K_1, K_2, and K_1K_2 are Galois over F.
(c) Show that $\mathrm{Gal}(K_1K_2/F)$ is the Klein 4-group. List the automorphisms in this group and their action on the roots in (a).
(d) Let L be a splitting field of f. Show that $\mathrm{Gal}(L/\mathbb{Q}) \cong D_8$ is isomorphic to the dihedral group of order 8. Make a nice field diagram and corresponding subgroup diagram showing the Galois correspondence for each subfield and corresponding subgroup.

Problem 4.2. Let $K = \mathbb{Q}(\sqrt{2+\sqrt{2}})$. Show that K is Galois over $\mathbb{Q}$ and that $\mathrm{Gal}(K/\mathbb{Q})$ is a cyclic group.

Problem 4.3 (M3-2). Let p be an odd prime, and let $\zeta = \zeta_p \in \mathbb{C}$ be a primitive pth root of unity and let $K = \mathbb{Q}(\zeta)$.

(a) Show that K is Galois and that the map

$$\begin{aligned} (\mathbb{Z}/p\mathbb{Z})^\times &\to \mathrm{Gal}(K/\mathbb{Q}) = G \\ a &\mapsto (\zeta \mapsto \zeta^a) \end{aligned}$$

is an isomorphism of groups.

Now let $H \leq G$ be the subgroup of index 2 in G. Let

$$\alpha = \sum_{\sigma \in H} \sigma(\zeta), \quad \beta = \sum_{\sigma \in G \setminus H} \sigma(\zeta).$$

(b) Show that $\alpha, \beta \in K^H$, and that if $\sigma \in G \setminus H$, then σ swaps α and β. Conclude that α, β are the roots of the polynomial $X^2 + X + \alpha\beta \in \mathbb{Q}[X]$.
(c) Show that

$$4\alpha\beta = \begin{cases} p-1, & p \equiv 1 \pmod 4; \\ p+1, & p \equiv 3 \pmod 4. \end{cases}$$

and conclude that the fixed field $K^H = \mathbb{Q}(\sqrt{\pm p})$, accordingly. *[Hint: Make sense of the 'solution' to* M3-2 *on page 128–129 and write it out fully.]*

Problem 4.4 (M3-4). Let K/F be a finite Galois extension with Galois group G and let $M = K^H$ with $H \leq G$. Show that $\mathrm{Aut}_F(M) = N/H$ where $N = N_G(H)$ is the normalizer of H in G.

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