MATH 81/111: RINGS AND FIELDS
HOMEWORK #2

Problem 2.1. Let R be a domain containing $\mathbb{C}$, and suppose that R is a finite-dimensional $\mathbb{C}$-vector space. Show that $R = \mathbb{C}$.

Problem 2.2.

(a) Compute the minimal polynomial of $\alpha = 2\cos(2\pi/5)$.
(b) Conclude that the regular 5-gon is constructible by straightedge and compass, express $\cos(2\pi/5)$ and $\sin(2\pi/5)$ using only square roots, and maybe construct a 5-gon by hand.

Problem 2.3. Prove that $e = \sum_{n=0}^{\infty} 1/n!$ is irrational, as follows.

(a) Show that $0 < N! \cdot (e - s_N) < 1/N$ where $s_N = \sum_{n=0}^{N} 1/n!$.
(b) Suppose that $e = p/q \in \mathbb{Q}$, and using (a) derive a contradiction.

(It is much harder to show that e is transcendental!)

Problem 2.4. Determine the splitting field and its degree over $\mathbb{Q}$ for the polynomials $X^4 + X^2 + 1$ and $X^5 - 4$.

Problem 2.5. Let F be a field with char $F \neq 2$, and let $L \supset F$ be a field extension of degree 4.

(a) Prove that there exists an intermediate field $L \supsetneq K \supsetneq F$ if and only if $L = F(\alpha)$ with α having minimal polynomial over F of the form $X^4 + aX^2 + b$ with $a, b \in F$.
(b) Suppose that $f(X) = X^4 + aX^2 + b \in F[X]$ is irreducible in $F[X]$ and that $\sqrt{b} \in F$. Show that if $\alpha \in L$ is a root of f, then so is $\sqrt{b}/\alpha$, and deduce that $L = F(\alpha)$ is a splitting field for f.

Problem 2.6 (sorta M2-2). Let F be a field of characteristic p, with p prime, and let $f(X) = X^p - X - a$.

(a) Show that if $f(X)$ is reducible in $F[X]$, then it splits into distinct factors in $F[X]$. Conclude that if $f(X)$ is irreducible, then $K = F[X]/(f(X))$ is a splitting field for f.
(b) Conclude that $X^p - X - a$ is irreducible over $\mathbb{F}_p[X]$ for all $a \in \mathbb{F}_p^\times$ and that the fields $K = F[X]/(f(X))$ obtained are isomorphic for all a.

Date: 14 January 2015; due Thursday, 22 January 2015.