# MATH 81/111: RINGS AND FIELDS
# HOMEWORK #1

**Problem 1.1**. Show that the polynomial $f(x) = x^3 - x^2 + 1$ is irreducible over $\mathbb{Q}$, and express in radicals a solution to the equation $f(x) = 0$.

**Problem 1.2 (M1-1)**. Let $F = \mathbb{Q}(\alpha)$ where $\alpha^3 - \alpha^2 + \alpha + 2 = 0$. Express $(\alpha^2+\alpha+1)(\alpha^2-\alpha)$ and $(\alpha-1)^{-1}$ in the form $a\alpha^2 + b\alpha + c$ with $a, b, c \in \mathbb{Q}$.

**Problem 1.3**. Let $D \in \mathbb{Z} \setminus \{0\}$ be squarefree and let $F = \mathbb{Q}(\sqrt{D})$. Let $\alpha = a + b\sqrt{D} \in F$.

(a) Show that the "multiplication by $\alpha$" map
$$\begin{aligned} \phi : F &\to F \\ \beta &\mapsto \phi(\beta) = \alpha\beta \end{aligned}$$
is a linear transformation of vector spaces over $\mathbb{Q}$.

(b) Compute the matrix of $\phi$ on the basis $1, \sqrt{D}$ of $F$.

**Problem 1.4 (sorta M1-3)**. Let $F$ be a field and $f(X) \in F[X]$.

(a) Show that $f(X)$ can have at most $\deg f$ roots.

(b) Let $G$ be a finite abelian group. Show that $G$ is cyclic if and only if $G$ has at most $m$ elements of order dividing $m$ for each divisor $m$ of $\#G$.

(c) Deduce that if $H \subseteq F^\times$ is a finite subgroup, then $H$ is cyclic.

**Problem 1.5 (sorta M1-2)**. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(a) Determine $[F : \mathbb{Q}]$.

(b) Find an element $\alpha \in F$ such that $F = \mathbb{Q}(\alpha)$ and compute its minimal polynomial.

**Problem 1.6**. Let $F$ be a field. Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.

*Date*: 7 January 2015; due Wednesday, 14 January 2015.