MATH 81/111: RINGS AND FIELDS
FINAL EXAM

Name ______________________________

Problem	Score	Problem	Score
1		5	
2		6	
3		7	
4			

Total ____________

Date: 13 March 2015.

Problem 1. Let $f(X) = (X^4 - 3)(X^2 - 2)$.

(a) Exhibit a splitting field for f.

(b) Give a presentation (in terms of generators and relations) for the Galois group $\mathrm{Gal}(f)$ and an embedding of $\mathrm{Gal}(f) \hookrightarrow S_6$.

Problem 2. Let K/F be a finite Galois extension with Galois group $G = \mathrm{Gal}(K/F)$, and let L/F be a finite extension of degree m with $\gcd(m, \#G) = 1$. Show that KL/L is Galois with $\mathrm{Gal}(KL/L) \cong G$.

Problem 3. Let F be a field. We say that $\beta \in F$ *can be written as a sum of squares* in F if there exist $\alpha_1, \dots, \alpha_n \in F$ such that

$$\alpha_1^2 + \cdots + \alpha_n^2 = \beta.$$

Let F be a finite extension of $\mathbb{Q}$ of odd degree. Show that -1 is not a sum of squares in F.

Problem 4.

(a) Let G be a group, let $H \leq G$ be a subgroup, and let

$$N = \bigcap_{g \in G} gHg^{-1}.$$

Show that $N \trianglelefteq G$ is the largest normal subgroup of G contained in H.

(b) Let K/F be a Galois extension with Galois group $G = \mathrm{Gal}(K/F)$. Let $F \subseteq M \subseteq K$ be an intermediate extension, corresponding to $H \leq G$. Let N be as in (a). Show that the fixed field of N is the *Galois closure* of M in K, i.e., the smallest extension of M that is Galois over F.

Problem 5. Show that a regular 9-gon is not constructible by straightedge and compass.

Problem 6.

(a) Give an explicit construction of $\mathbb{F}_4$.

(b) Is the polynomial $f(X) = X^4 + X + T$ separable over $\mathbb{F}_4(T)$?

(c) The polynomial $f(X) = X^4 + X + T$ is irreducible over $\mathbb{F}_4(T)$. Compute the Galois group of f over $\mathbb{F}_4(T)$.

Problem 7. Let p be prime and let F be a field in which $X^p - 1$ splits into distinct linear factors. Let $a \in F^\times \setminus F^{\times p}$, and let $K = F(\sqrt[p]{a}) = F[X]/(X^p - a)$. Show that the polynomial $X^p - b \in F[X]$ splits in K if and only if $b = a^j c^p$ for some $c \in F^\times$ and $j \in \{0, \ldots, p-1\}$.