Ordered Sets

Ken Bogart

November 27, 2001

The main activity of this course will be solving and discussing solutions of problems. These notes contain the problems and the bare essential preparatory and connective material to prepare the reader for the problems and connect the problems together. Most of the preparatory and connective material will be discussed in class, usually in more detail, as we work our way through the problems. Sometimes problems will be given as soon as they can be stated, even though later problems might make them more accessible. This is intentional; think how good you will feel if you solve them without needing the extra stuff.

Students will keep a problem book in which they write up solutions to the problems. Students will receive extra credit for multiple solutions to problems. Students are expected to rework problems until they have (if possible) at least one essentially correct solution to each problem. The problem book will begin with a table of contents which gives the page number on which each attempt at a problem is made and gives a place for a grade on each attempt. Problems will be graded with an @ sign (which means "again," as in "do this problem again"), .5, which means "I'll give you half credit for this problem if you want to stop here," .9, which means "essentially correct but needs optional cosmetic work to become a 1," or 1, which should be self explanatory. Multiple solutions to problems should have multiple entries in the contents so that they can receive multiple credit. When multiple solutions surprise me (pleasantly) I will give double credit for the ones that surprise me. Students should make sure they maintain a balance between trying to redo "old" exercises and continuing to attempt new problems.

These notes will be updated at more or less random times during the term.

1. Consider the relationships of

   (a) Less than or equal to on the integers

   (b) Subset of or equal to on the subsets of a set

   (c) Divides (is a factor of) on the positive integers

   (d) Lies entirely to the left of or equals on the closed intervals along the x axis.[1]

   Find as many common properties of these relationships as you can.

A reflexive, antisymmetric, transitive relation P on a set X is called an (partial) *order* or *ordering* of X. The pair (X, P) is called an (partially) *ordered set* or a *poset*. Thus the relations in Problem 1 are all examples of partial orderings. We write $x \preceq y$ in P for $(x, y) \in P$ [recall, a relation is a set of ordered pairs] and $x \prec y$ in P for $x \neq y$ and $(x, y) \in P$. We leave out the phrase "in P" whenever P is clear from context. We can visualize an ordering P of a set X in a number of ways; two are the digraph and the diagram of the ordering. We draw a *digraph* of P by drawing

[1] For example, $[1, 3]$ lies entirely to the left of $[4.5, 6]$ but $[1, 3]$ does not lie entirely to the left of $[2, 5]$.

a vertex (dot) for each element of X and drawing an arrow from x to y if $x \preceq y$ in P. Drawing the digraph of the less than or equal to relation of the set $\{1,2,3,4,5\}$ should convince you that the digraph can often be a sloppy way of visualizing an order relationship. On the other hand, in Figure 1, you see a clean way to visualize the less than relation on the set $\{1,2,3,4,5\}$.

Figure 1: Visualizing the less than relation on the set $\{1,2,3,4,5\}$.

We will introduce a few ideas that will help us describe a drawing similar to that in Figure 1 for any finite ordered set. Two elements of X are *comparable* in P if $x \preceq y$ or $y \preceq x$ in P. Otherwise they are *incomparable*. A *chain* is a set of mutually comparable elements, and an *antichain* is a set of mutually incomparable elements. We say x is *below* y and y is *above* x if $x \prec y$ in P. We say y *covers* x if $x \prec y$ in P and there are no elements that are above x but below y. The *height* of an element x is the maximum number of elements in a chain of elements below x (or is infinite if no maximum exists).

We draw a *diagram* of (X,P) by drawing a horizontal row of dots, one for each element of height 0, then a higher horizontal row of dots, one for each element of height one, and so on. Then we draw a line segment from the dot for y to the dot for x if y covers x. When it is applied to $\{1,2,3,4,5\}$ with the usual ordering, this process gives a diagram such as we saw in Figure 1.

2. Draw a diagram of the following ordered sets.
   (a) The subsets of $\{1,2,3\}$ ordered by set inclusion.
   (b) The divisors of 30 ordered by $x \preceq y$ if x divides y.
   (c) The divisors of 36 ordered by $x \preceq y$ if x divides y.
   (d) The closed intervals with integer endpoints between 0 and 3 (don't forget one point intervals) ordered by $[a,b] \prec [c,d]$ if $b < c$.

3. The definition of isomorphism for ordered sets is tricky. Give an example of two ordered sets (X,P) and (Y,Q) that are "obviously nonisomorphic" with a bijection $f : X \to Y$ such that if $x \preceq y$, then $f(x) \preceq f(y)$. Figure out what a good definition of isomorphism is.

4. Apply your good definition of isomorphism to explain the diagrams you got in parts a and b of Problem 2. Don't restrict yourself to three-element sets, though.

5. An *upper bound* to a set S of elements of an ordered set (X,P) is an element of X above or equal to every element in S. A *least upper bound* to S is an upper bound which is below or equal to every upper bound of S. (Lower bounds and greatest lower bounds are defined siminlarly.) Need a set of elements in an ordered set have an upper bound? If a set has an upper bound, need it have a least upper bound? An ordered set is called a *lattice* if each pair of elements has a greatest lower bound and a least upper bound. (For future reference, the standard notation for the greatest lower bound for x and y is $x \wedge y$ and for the least upper bound is $x \vee y$. These are read "x meet y" and "x join y" respectively.) Which of the examples in Problem 1 and Problem 2 are lattices?

An ordering L of X is called *linear* if X is a chain of L, i.e. all elements of X are comparable. An ordering P on X is called an *extension* of an ordering Q on X if $Q \subseteq P$ (as sets of ordered pairs). Suppose P is an ordering of a finite set X. Make a list of the elements of X by listing all the elements of height 0 and then listing all the elements of height 1, and so on. Define an ordering L by $x \prec y$ in L if x comes before y in the list. (It is clear that this gives a linear ordering, isn't it?) This shows that every ordering of a finite set has a linear extension. In the next set of problems, if you aren't familiar with Zorn's lemma, transfinite induction, the axiom of choice or some other way of dealing with infinite sets, you may assume X is finite.

6. Prove or give a counter-example. Suppose P is an ordering of X. If x and y are incomparable in P, then there is a linear extension L of P such that $x \preceq y$ in L.

7. Prove or give a counter-example. Every ordering is the intersection of all its linear extensions.

8. An ordering P of X has dimension d if it is an intersection of d, but no fewer than d, linear orderings of X. What is the dimension of a chain? An antichain? The subsets of $\{1, 2, 3\}$ with the inclusion ordering? The divisors of 36 with the divisor ordering?

9. Find a three-dimensional ordering on a six-element set. Find an n-dimensional ordering on a $2n$-element set.

If P is an order on X, and Q is an order on Y, then the *product* $P \times Q$ of P and Q is the order on the Cartesian product $X \times Y$ given by $(x_1, y_1) \preceq (x_2, y_2)$ iff $x_1 \preceq x_2$ and $y_1 \preceq y_2$. (Note: we should have said $(x_1, y_1) \preceq (x_2, y_2)$ in $P \times Q$ iff $x_1 \preceq x_2$ in P and $y_1 \preceq y_2$ in Q; however when confusion is unlikely, we omit the modifier "in. . . .")

10. Express the "divides" ordering on the divisors of 36 as a product of simpler orderings. Express the "subset of" ordering on the subsets of $\{1, 2, 3\}$ as a product of simpler orderings.

11. Show that if an ordering has dimension d, then it is isomorphic to a restriction of a product of d linear orderings. (The *restriction* of an ordering P of X to a subset Y is $P \cap (Y \times Y)$. It is routine to verify that the restriction of an ordering of X to any subset of X is an ordering of that subset. People often say (Y, Q) is a *subposet* of (X, P) when $Y \subseteq X$ and Q is the restriction of P to Y.)

12. Show that if an ordering is isomorphic to a restriction of a product of d linear orderings, then it has dimension at most d. (Revisit this after doing the next couple exercises if you can't see a way to do it now.)

We think of a linear order of a set X as lining the elements of the set up in a line. We can make this precise as follows.

Theorem 1 *If L is a linear ordering of a finite set X, then there is a one-to-one function $f : X \to Z$ (with Z the integers) so that $x \prec y$ in L if and only if $f(x) < f(y)$.*

Proof. Clearly the theorem is true when $|X| = 1$, so assume inductively that the theorem is true when $|X| = n$. Now let $|X| = n + 1$. Choose an element w of X under as few other elements of X as possible. If $w \prec z$, then every element above z is above w by transitivity. This contradicts our choice of w. Therefore w has no elements above it, and so by linearity, all other elements of X below it. The restriction L' of L to $X - \{w\}$ is linear, so by our inductive assumption, there is a one-to-one function $g : (X - \{w\}) \to Z$ such that $x < y$ in L' if and only if $g(x) < g(y)$. Since X is finite, g has a maximum value M. Let $f(w) = M + 1$, and $f(x) = g(x)$ for $x \in X - \{w\}$. Now $x \prec w$ if and only if $f(x) < f(w)$, and otherwise $f(x) \prec f(y)$ if and only if $f(x) < f(y)$ because $f = g$ on $X - \{w\}$. Clearly f is one-to-one by construction. Therefore by the principle of mathematical induction the theorem holds for all finite sets. ∎

The element w we chose in the proof is called a *maximum* or *top* element, because it is above everything else. A slightly weaker is the idea of a *maximal* element; namely an element that has nothing above it. (Minimal elements are similarly defined.) Do you see the argument that shows that every finite ordered set has at least one maximal (minimal) element?

The defining condition of a linear order has a natural expression in terms of the "sum" of two ordered sets. The ordered set (X, P) is said to be ths *sum* of the ordered sets (X_1, P_2) and (X_2, P_2) if X is the disjoint union of X_1 and X_2, and P is the (disjoint) union of P_1 and P_2. (Why did I parenthesize disjoint?) We write $(X_1, P_1) + (X_2, P_2)$ for the sum of the two ordered sets, and when we use this notation for arbitrary posets in which X_1 and X_2 are not necessarily disjoint, we mean to use isomorphic copies of the two posets on disjoint sets. We use the notation $\underline{1}$ to stand for the ordered set with just one element. Then $\underline{1} + \underline{1}$ stands for a two element antichain. In this notation, an ordering P of X is linear if and only if it has no restriction isomorphic to $\underline{1}+\underline{1}$. We use the notation $\underline{2}$ to stand for a two-element ordered set in which the elements are comparable. For reasons that may be apparent in the next exercise, we say that an order is "weakly linear" (which most people shorten to *weak*, as we will) if it has no restriction isomorphic to $\underline{2} + \underline{1}$.

13. Show that an order P of a set X is a weak order if and only if there is a function $f : X \to Z$ such that $x \prec y$ in P if and only if $f(x) < f(y)$. Give an intuitive description of a weak ordering.

14. An ordering has *weak dimension* d if it is an intersection of d and no fewer weak orderings. Show that an non-weak ordering has dimension d if and only if it has weak dimension d.

15. What is the dimension of the ordered set of all subsets of an n-element set?

16. What is the dimension of a product of k chains, each of length at least two?

17. Give the best upper bound you can on the dimension of an ordered set which is a union of k chains. Show it is best possible.

18. What is the best upper bound you can give for the dimension of an ordered set whose largest antichain has size w? The size of the largest antichain in an ordered set is called its *width*; that is why I used w.

19. For a set X ordered by P, find the strongest relationship you can between the minimum number of chains whose union is X and the maximum number of elements in an anitchain (if either of these numbers is finite). This is known as Dilworth's theorem. Prove it.

20. A lattice L is called distributive if for each x, y and z in L,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Give two examples of distributive lattices.

21. (This exercise is one the students came up with in class, and they liked it, so I'm including it for you.) Find the lattices with five or fewer elements and determine which are distributive.

22. Show that the subgroups of a group form a lattice with the usual ordering. Is it distributive? Show that the subspaces of a vector space form a lattice (with the subset ordering). Is it distributive?

23. Can you say anything interesting about the dimension of a finite distributive lattice?

24. Is a product of distributive lattices distributive?

25. An element of a lattice is said to be *join-irreducible* if it cannot be written as a join of elements different from it. The term *meet-irreducible* is defined similarly. Are there any join- or meet-irreducible elements in the lattice of subsets of a finite set? What about in the lattice of subsets of the integers? What about in the integers ordered by the usual less than relation? What about the lattice of divisors of a positive integer? What about the lattice of all positive integers ordered by division?

26. In how many different ways can you write an element of a finite distributive lattice as a join of join-irreducible elements?

27. If someone tells you what the poset of join irreducible elements of a finite distributive lattice is, just how much are they telling you about the lattice?

28. Can you say anything more interesting about the dimension of a distributive lattice than you could before?

29. Is a distributive lattice a lattice of some of the subsets of a set? A *sublattice* of a lattice L is a subset of L which is a lattice with the same meet and join operations as L. In particular is a distributive lattice a sublattice of the lattice of all subsets of a set?

30. A sublattice of a lattice L is a subset of L which is a lattice with the same meet and join operations as L. When is a lattice a sublattice of a product of chains?

31. Can you say anything more interesting about the dimension of a distributive lattice now?

32. Let D_5 stand for the 5-element nondistributive lattice of height 2 and M_5 stand for the 5-element nondistributive lattice of height 3. Can you find a group whose subgroup lattice has a sublattice isomorphic to D_5? To M_5?

33. Find three elements A, B, and C in M_5 such that

$$A \wedge (B \vee C) \neq (A \wedge B) \vee (A \wedge C) = B \vee (A \wedge C).$$

Can you find three such elements in the lattice of submodules of a module? (Notice that this includes as special cases the lattice of subspaces of a vector space and the lattice of subgroups of an abelian group.)

34. A lattice is *modular* if, whenever $b \preceq a$, then $a \wedge (b \vee c) = b \vee (a \wedge c)$. In what sense is modularity like a limited amount of distributivity? Show that the lattice of submodules of a module is modular.

35. Classify the lattices with five or fewer elements as modular or not.

36. Show that the normal subgroups of a group form a modular lattice.

37. The diamond isomorphism theorem for groups tells us that in the lattice of normal subgroups of a group, the interval $[x \wedge y, x] = \{z : x \wedge y \leq z \leq x\}$ from $x \wedge y$ to x is isomorphic to the interval from y to $y \vee x$. Show that a lattice is modular if and only if for each x and y the interval from $x \wedge y$ to x is isomorphic to the interval from y to $y \vee x$. Explain why this means that for each x and y the "meet with x map" from $[y, y \vee x]$ to $[x \wedge y, x]$ is inverse to "join with y map" from $[x \wedge y, x]$ to $[y, x \vee y]$. Thus if a lattice is not modular, there are elements x and y such that the composition of these two maps is not the identity.

38. What do you think you can say about a lattice that has no sublattice isomorphic to M_5? Can you prove it?

39. What do you think you can say about a lattice that has no sublattice isomorphic to either M_5 or D_5? Can you prove it?

40. An apparently weaker condition than the necessary and sufficient condition of Exercise 37 is that if x covers $x \wedge y$, then $x \vee y$ covers y. Is it really weaker? In other words, if a (finite) lattice satisfies this condition is it modular, or is there an example of a lattice that satisfies this condition for every x and y and yet is not modular?

41. The *dual* (reverse would be a better word, but dual appears in the literature) of an order on X is the order in which every ordered pair is reversed. Is the dual of a distributive lattice distributive? Is the dual of a modular lattice modular?

42. With distributive lattices you were able to develop a theory of "factoring" an element into a meet or join of irreducible elements. It was very helpful to be able to think in terms of a paradigm of subset lattices and products of chains as cannonical examples of distributive lattices in ordeer to see what that theory should be. With modular lattices our paradigms are not so clear, but there are some, namely the lattice of subspaces of a vector space and the lattice of ideal of a ring. (And, more generally, the lattice of submodules of a module.) Using these as paradigms, what can you figure out about "factoring" elements of a modular lattice into join- or meet- irreducible elements?

43. A *bipartite graph* is a graph whose vertex set is a union of two disjoint sets X and Y with all edges connnecting an element of X to an element of Y. A *vertex cover* of the edges is a set of vertices such that every edge includes at least one member of that set. A *matching* is a set of edges no two of which have a vertex in common. Draw an example of a bipartite graph with 10 edges and a matching of size five. For this example, write down a vertex cover of the edges. How do you think the size of a minimum sized vertex cover of the edges and a maximum sized matching are related in a bipartite graph? Prove you are right. With any luck, you've stated and proved König's Theorem. Hopefully your proof has something to do with ordered sets.

44. The concepts of vertex cover of the edges and matching make sense in all graphs. Explore the truth or falisty of your theorem in the previous exercise for graphs in general.

45. In a bipartite graph with vertex xets X and Y, for any subset S of X or Y we define $R(S)$ (the relatives of S) to be the set of all vertices which are an endpoint of an edge whose other endpoint is in S. A matching which has an edge containing each vertex of X is called a matching from X into Y. Draw an example of a bipartite graph on a set X of size 5 and a set Y with a matching from X into Y. In any bipartite graph, if you have a matching from X into Y, what can you say about the size of S and $R(S)$ for each subset S of X? Think of the most beautiful conjecture you could make, make it, and prove it. With any luck you have stated "Hall's Marriage Theorem," also known as the König-Hall Theorem.

46. Given a set X ordered by P, the *Fulkerson graph* (my terminology, not well known) of (X, P) is the bipartite graph with two vertex sets X_1 and X_2, and a bijection $x \to x_i$ from X into X_i so that there is an edge from x_1 to y_2 iff $x < y$ in P. Suppose you know König's theorem from Problem 43. Use the Fulkerson graph to derive Dilworth's theorem.

47. Suppose X is a set ordered by P. Given a set $\mathcal{C}$ of chains covering X, if there is a chain C_1 whose top element is below the bottom element of a chain C_2, then we may reduce the number of chains in the covering by taking the union of C_1 and C_2. Give an example of on ordered set and a chain covering so that repeating this process until there are no such C_1 and C_2 does *not* give a covering with a minimum number of chains. Suppose instead that there are chains C_1, C_2, and C_3 in $\mathcal{C}$ such that the top element of C_1 is below some element x of C_2, but not below elements lower than x, and all elements below x are below the bottom element of C_3. How does this let you reduce the number of chains in the chain covering? Does repeating the first process and then this process until repetition is impossible give a minimum sized chain decomposition? If so, prove it, and if not, give a counter-example and find a process that does work.

48. We count the people in a room, and then we ask "How many people have sisters?" Then we ask "How many people have two or more sisters?" To belabor the obvious, how do we find out the number of people with no sisters, with exactly one sister? Inclusion-exclusion counting is quite similar. Suppose we are trying to compute the number of permutations of $\{1, 2, 3\}$ with no fixed points. We ask how many permutations are there? (Play along and answer the questions as we go.) How many permutations have 1 as a fixed point? How many have 2 as a fixed point? How many have three as a fixed point? How many have 1 and 2 as fixed points? 1 and 3? 2 and 3? How many have 1,2, and 3 as fixed points? Now we ask how many permutations have 1 and 2 and only one and two as fixed points? How many have 3 and only 3 as fixed points? How many have no fixed points? In both these examples, we have 2 functions f_a and f_e (standing for "f-sub **at least**" and "f-sub **exactly**") defined on an ordered set $(S, \preceq)$. The two functions are related by the system of equations

$$f_a(x) = \sum_{y: x \preceq y} f_e(y). \tag{1}$$

In both cases, it turned out that we knew the numbers $f_a(x)$. We had $|S|$ equations in $|S|$ unknowns (in the first case S is $\{0, 1, 2\}$ and in the second case, S is the power set of $\{1, 2, 3\}$) and as luck would have it, we can solve the system of equations for $f_e(x)$ for every element x of S. The main question of this problem is "Was it just luck?" In other words, suppose we have an ordered set $(S, \preceq)$, and we have two functions related by the system of equations 1.

Can we always solve the equations for f_e in terms of f_a? In each case, take a linear extension of $(S, \preceq)$, and write down the system of equations so that the equation for $f_a(x)$ precedes the equation for $f_a(y)$ if and only if x precedes y in the linear extension. Think of this system of equations as a matrix equation. What is special about the matrix of coefficients? Was it just luck that we were able to solve the systems for the unknowns $f_e(x)$? What if I have the system of equations defined by 1 for some arbitrary known function f_a? Is it possible to solve for the values of the function f_e?

49. Let $f_e(k)$ be the number of generators of a cyclic group of order k. What is

$$f_a(n) = \sum_{k:k|n} f_e(k)?$$

(Think of f_a here as standing for "f-sub **at most**.") For a fixed m, determine whether the system of equations we get by considering all divisors n of m can be solved to give the unknowns $f_e(k)$ for all the divisors k of m.

Solving a system of equations is, in practice, often different from determining whether such a solution is guaranteed to exist. We are going to analyze the process of solving a system of equations of the form 1. One tool we could use in this analysis is matrix algebra. The subscripts that are involved in dealing with matrices would, however, slow us down, and they obscure patterns involving ordered sets. Thus we will use an approach which parallels matrix algebra, but is more faithful to the structure of our underlying ordered set. It is good for intution's sake to translate our results back into the terminology of matrix algebra as we go. For a partially ordered set $\mathbf{P} = (X, \preceq)$ are going to introduce an ring called the Incidence Algebra of $\mathbf{P}$, and a module over that ring called the Möbius Module of $\mathbf{P}$. For an n-element poset, the ring is isomorphic to a subalgebra of the algebra of n by n matrices, and the module is isomorphic to the vector space of column vectors of length n. The action of the ring on the module is analogous to the multiplication of column vectors by matrices.

Choose a field F. The Möbius Module (my terminology, not widely used) of $\mathbf{P}$ over F is simply the set of functions from X to F. (Remember, $\mathbf{P} = (X, \preceq)$.) We add two members of the module in the natural way, namely $(f+g)(x) = f(x) + g(x)$. The *incidence algebra* $\mathcal{I}(\mathbf{P})$ of $\mathbf{P}$ consists of all functions φ of two variables with the property that $\varphi(x,y)$ is zero unless $x \preceq y$. The product in the ring is defined by

$$\varphi \circ \psi(x,y) = \begin{cases} \displaystyle\sum_{z:x\preceq z\preceq y} \varphi(x,z)\psi(z,y) = \sum_{z=x}^{y} \varphi(x,z)\psi(z,y) & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

We define the addition in the ring in the natural way (what is it?). Notice that for the product we have defined to make sense, we need to require that every interval $[x,y]$ of our ordered set is finite. (An ordered set is called *locally finite* if its intervals are all finite.) We make the Möbius Module into a (right) module over the incidence algebra by defining $\varphi \circ f$ by

$$(\varphi \circ f)(x) = \sum_{y:x\preceq y} \varphi(x,y)f(y).$$

(This is actually the "upper Möbius module" of (X, P); the same set is also a "lower Möbuis module" of (X, P) which is a left $\mathcal{I}(\mathbf{P})$ module. If you are having fun with this, figure out what the action is.) Notice that for this to make sense, we need a stronger "finiteness" condition than simply that all intervals are finite. (What is the weakest useful condition yu can think of?) For now, when we speak of the Möbius Module of $\mathbf{P}$, we will assume $\mathbf{P}$ is finite.

50. Prove that the product we have defined in the incidence algebra is associative. What function δ is the multiplicative identity?

51. When does an element of the incidence algebra have a multiplicative inverse? In particular, does the element ζ (zeta) given by

$$\zeta(x,y) = \begin{cases} 1 \text{ if } x \preceq y \\ 0 \text{ otherwise} \end{cases}$$

have an inverse?

52. The elements ζ and δ in the incidence algebra have been described above. What does $(\zeta - \delta)^n(x, y)$ count?

53. The inverse of the zeta function is called the Möbius function, denoted by μ. What is $\mu(x, x)$, for any x in an ordered set $\mathbf{P}$? If y covers x in $\mathbf{P}$, what is $\mu(x, y)$? If $x \prec y$ (which means $x \preceq y$ and $x \neq y$), what is $\mu \circ \zeta(x, y)$? What is $\zeta \circ \mu(x, y)$? What is $\sum_{z=x}^{y} \mu(x, z) = \sum_{z:x \preceq z \preceq y} \mu(x, z)$? What is $\sum_{z=x}^{y} \mu(z, y)$?

54. What is $\mu(X, Y)$ in the ordered set of subsets of a set S if $X \subseteq Y$? What if $X \not\subseteq Y$?

55. Suppose now that $\mathbf{P}$ is a finite ordered set. Recall that if φ is in the incidence algebra and f is in the Möbius Module, we define $\varphi \circ f(x) = \sum_{y:x \preceq y} \phi(x, y) f(y)$. Let S be a finite set of objects, let R be a set of properties that elements of S may or may not have, and let $\mathbf{P}$ be the ordered set of subsets of R, ordered by set inclusion. Suppose that for each subset X of R, $f_e(X)$ is the number of elements of S with exactly the properties in the set X. What does $\zeta \circ f_e(X)$ count? Suppose that

$$f_a(X) = \sum_{Y:X \subseteq Y} f_e(Y).$$

How can the values of f_e be computed from the values of f_a? You have just stated the "Principle of Inclusion and Exclusion."

56. (Rota's Fundamental Theorem of Möbius Inversion.) Let f be a function defined on an ordered set $\mathbf{P}$, and let $g(x) = \sum_{y:x \preceq y} f(y)$. How can we use μ to compute f from g?

57. Explain how The Principle of Inclusion and Exclusion is a special case of Möbius Inversion.

58. If ζ_1 and ζ_2 are the zeta functions of (X_1, P_1) and (X_2, P_2), and ζ is the zeta function of $P_1 \times P_2$, how can $\zeta((x_1, y_1), (x_2, y_2))$ be computed from values of ζ_1 and ζ_2?

59. If φ_1 and φ_2 are members of $\mathcal{I}(P_1)$ and $\mathcal{I}(P_2)$, define $\varphi_1 \cdot \varphi_2$ in $\mathcal{I}(P_1 \times P_2)$ by

$$\varphi_1 \cdot \varphi_2((x_1, x_2), (y_1, y_2)) = \varphi_1(x_1, y_1)\varphi_2(x_2, y_2)$$

Note that we are using a raised dot rather than a raised circle to denote this "product," and that the product takes elements of $\mathcal{I}(P_1)$ and $\mathcal{I}(P_2)$ and gives us an element of $\mathcal{I}(P_1 \times P_2)$. By the way, how do we know that this product lies in the incidence algebra of $P_1 \times P_2$? Express your answer about the zeta function of the product in the previous problem in this notation. What is the Möbius function of the product? Prove your answer is correct. What about the Möbius function of the product of a finite number of ordered sets?

60. What is the Möbius function of the ordered set of 3-tuples of zeros and ones with the product ordering? What does this have to do with the Möbius function of the ordered set of subsets of a set? What is the Möbius function of the ordered set of divisors of 180, ordered by divisibility?

61. Solve the system of equations you gave in problem 49.

In a graph, a sequence $x_0e_1x_1e_2 \cdots e_nx_n$ of vertices and edges is called a *walk* from x_0 to x_n if each e_i is the edge $\{x_{i-1}, x_i\}$. If no edges or vertices are repeated in the sequence, a walk is called a *path*. We say x is connected to y if there is a walk starting at x and ending at y. It is straightforward to show that the relation of being connected is an equivalence relation. The equivalence classes of this relation are called *connected components*, and a graph is called *connected* if it has one connected component.

The *partition lattice* on n elments is the set of partitions of an n-element set with $P \preceq \mathbf{Q}$ if every class of $\mathbf{P}$ is a subset of some block of $\mathbf{Q}$. I forgot earlier to ask you to show this is an example of a lattice!

62. How many graphs are there with vertex set $\{a, b, c, d, e, f, g\}$? How many graphs have a connected component partition that is a refinement of $\{a, c, g\}$, $\{b, e\}$, $\{d, f\}$? If $f_e(\mathbf{R})$ is the number of graphs whose connected component partiton is exactly $\mathbf{R}$, write down a system of equations that will allow you (at least in principle) to compute $f_e(\mathbf{R})$ for every partition $\mathbf{R}$ of the vertex set. If $\mathbf{R}$ is the partition with one (equivalence) class, what does $f_e(\mathbf{R})$ count? Express $f_e(\mathbf{R})$ in terms of the Möbius function of an appropriate ordered set. Since we do not yet know this Möbius function, this is all we can do for now. We will come back to this problem later.

63. As you may know, "coloring graphs" is a colorful chapter of the history of combinatorial mathematics. A coloring of a graph is an assignment of colors to its vertices, i.e., a function from the vertex set to some set K of colors. (We use K so C can stand for class, as in equivalence class.) A coloring is called proper if no two vertices joined by an edge are given the same color. (The four color theorem asserts that a graph that can be drawn in the plane with no crossings among the edges can be properly colored in 4 colors.) We are interested here in the number of proper colorings of a graph G in k colors. It is probably not particularly surprising that we count proper colorings by thinking first about all colorings, subtract out those that give some two adjacent vertices the same color, add back in .... However we will attack the problem in a more organized way. A partition of the vertex set of a graph is called

a *bond* if, for each class C of the partition, each two vertices in C are connected by a path all of whose vertices are in C. Explain why any bond is a refinement of the connected component partition of the graph. The bonds of a graph are ordered by the refinement ordering for partitions. Consider the graph on the vertices $\{1,2,3,4\}$ that has edges $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{4,1\}$,and $\{1,3\}$. Find a partition that is not a bond and draw the bond lattice of this graph.

64. A coloring of a graph gives a bond as follows. We say two vertices are equivalent if there is a walk, all of whose vertices have the same color, joining them. Explain why this is an equivalence relation and why the equivalence class partition of this equivalence relation is a bond. What is the bond of a proper coloring? For a given bond $\mathbf{B}$ of G, how many colorings of G are there whose bond is a coarser than (or equal to) $\mathbf{B}$? Use this information to find a formula (which could have a Möbius function in it) for the number of proper colorings of G with k colors. Notice that you have found a polynomial function of k; this polynomial is called the *Chromatic Polynomial* of the graph.

65. The complete graph on n vertices is the graph on n vertices which has all two element subsets of the vertex set as edges. What partitions are bonds of this graph, i.e. what is the ordered set of bonds of this graph? How big does k have to be for this graph to have any proper colorings with k colors? Give a formula for the number of proper colorings of this graph with $k \geq 0$ colors. Thinking of k as a variable, what is the coefficient of k in this polynomial?

66. Use $\mathbf{b}$ (for bottom) to stand for the partition of a set X into parts of size one, and use $\mathbf{t}$ (for top) to stand for the partition of a set X into one part. What is $\mu(\mathbf{b},\mathbf{t})$ in the ordered set of partitions of an n-element X? What is $\mu(\mathbf{P},\mathbf{t})$ for an arbitrary partition $\mathbf{P}$ of X into m classes?

67. Return to problem 62 with the Möbius function of problem 66, and convert your formula for the number of connected graphs on n vertices into a sum of known quantities over all partitions of the integer n.

68. Is the bond lattice of a four-cycle modular? What about the bond lattice of a complete graph on four vertices? Do they satisfy the condition that whenever x covers $x \wedge y$, then $x \vee y$ covers y?

69. If G is a graph and e is an edge of G with endpoints x and y, then the graph G/e, called the *contraction* of e from G and read as "G mod e" is the graph we get by identifying x and y and deleting e from our edge set. How is the bond lattice of G/e related to the bond lattice of G?

70. A lattice is called *(upper) semimodular* if whenever x covers $x \wedge y$ then $x \vee y$ covers y. Prove that the bond lattice of a graph is semimodular. An *atom* of a lattice (with a bottom element) is an element which covers the bottom element. Explain why every element of of a bond lattice is a join of atoms. A*geometric lattice* is a semimodular lattice in which every element is a join of atoms, so you've just show that a bond lattice is geometric.

71. Show that in a semimodular lattice for each x and y, all saturated chains from x to y have the same number of elements. (*Saturated* means that there are no elements we can add to the chain without destroying its chainliness.)

72. Show that a semimodular lattice satisfies the "exchange property" which states that if a and b are atoms, and $a \prec b \vee x$ but $a \not\prec x$, then $b \prec a \vee x$.

73. Show that a lattice is geometric if and only if it satisfies the exchange property and every element is a join of atoms.

74. Show that if we take a finite set of points in Euclidean space and let L be the lattice of all (linear) subspaces they span. Show that L is geometric. Does the same remain true if we replace (linear) by (affine)?

75. (This is a proof discovered by Curtis Greene for a very useful lemma aobut Mobius functions. I can't figure out how to lead you to discover it, so it won't be as exciting as it should be. On the other hand, I'm going to make you figure out the Lemma, so there should still some fun.) Suppose L is a lattice. We are going to deal with formulas relating the Mobius function of L to that of an interval $[a, t]$ in L, where t stands for the top element. For $z \in [a, t]$, define

$$f(z) = \sum_{x : x \vee a = z} \mu(b, x).$$

Let

$$g(z) = \sum_{w : w \preceq z} f(w).$$

There is a lot you can say about the values of $g(z)$. Say what you can. What does this tell you about f? What does this tell you about μ?

76. Show that if x covers y in a geometric lattice with bottom element b, then $\mu(b, x)$ and $\mu(b, y)$ have opposite signs. (A good way to do this is to show that in a geometric lattice, $\mu(b, x)(-1)^{r(x)}$ is positive.) What does this tell you about the coefficients of the chromatic polynomial (See exercise 64) of a graph?

77. An order $(X, \preceq)$ is called an *interval order* if for each x in X we have a non-zero length closed interval $I_x = [a_x, b_x]$ of real numbers so that $x \prec y$ iff $b_x \leq a_y$. Such an assignment of intervals to elements of our ordered set is called an *interval representation*. In fact, so long as X is finite, the interval can be in a finite linearly ordered set rather than the real numbers. Explain why. Temporarily define a strong interval order to be an order (Y, α) such that we have an assignment of a (possibly length 0) intervals $J_x = [c_x, d_x]$ of real numbers to the elements of Y in such a way that $x \;\alpha\; y$ if and only if $d_x < c_y$, (i.e. the interval for x lies entirely to the left of the interval for y). Show that an order is an interval order if and only if it is a strong interval order.

78. Find all orders on four or five element sets that are not interval orders. Do you see a relationship among them? If not, find a few orders on a six element set that are not interval orders and look for relationships.

In an ordered set we define the *predecessor set* of an element x by

$$\mathrm{Pred}(x) = \{y | y \prec x\},$$

and the predecessor set of a set S by

$$\mathrm{Pred}(S) = \{y | y \prec x \text{ for all } x \in S\} = \bigcap_{x : x \in S} \mathrm{Pred}(x).$$

Successor sets of elements and sets are defined similarly.

79. Consider an ordered set that contains no restrictions isomorphic to any of the four element orders you found in Exercise 78. Show that the predecessor sets of single elements are linearly ordered by set inclusion. In the case of finite ordered sets, describe all predecessor sets of sets that are not predecessor sets of single elements. (Don't forget the empty set.) Explain why the set of all predecessor sets of sets is linearly ordered by set inclusion. Is there always an interval representation of your ordered set in this linearly ordered set?

80. An order is called a *semiorder* (or *unit* interval order) if it has an interval representation in which all intervals have the same length. An order is called a *proper* interval order if it has a representation in which no interval is a proper subset of any other. It is immediate from the definitions that a semiorder is proper. Why? Is it the case that a proper interval order is a semiorder?

81. Find all orders on four or five element sets that are not semiorders.

82. Consider an ordered set that has no restrictions isomorphic to any of the four element orders you found in Exercise 81. Is it a semiorder?