# Final Exam for Math 103
# Due Wednesday, December 10, 2008

Work on one side of $8\frac{1}{2} \times 11$ inch paper only. Start each problem on a separate page. (This last requirement can be waived for those LaTeX users whose very short and elegant solutions would result in an uncomfortable waste of paper.)

1. Let $m$ be Lebesgue measure on $\mathbf{R}$, let $\mathscr{L}$ be the $\sigma$-algebra of Lebesgue measurable sets, and let $\mathcal{B}_{\mathbf{R}}$ be the $\sigma$-algebra of Borel sets in $\mathbf{R}$. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable.

(a) Observe that the diagonal $\Delta := \{ (x, x) \in \mathbf{R}^2 : x \in \mathbf{R} \}$ is in $\mathcal{B}_{\mathbf{R}} \otimes \mathcal{B}_{\mathbf{R}} = \mathcal{B}_{\mathbf{R}^2}$.

(b) Show that the graph of $f$, $G(f) := \{ (x, f(x)) \in \mathbf{R}^2 : x \in \mathbf{R} \}$ is in $\mathscr{L} \otimes \mathscr{L}$.

(c) Explain why

$$\{ y \in \mathbf{R} : m\big(\{ x \in \mathbf{R} : f(x) = y \}\big) > 0 \}$$

has Lebesgue measure zero.

(Personally, I find the conclusion of part (c) interesting even when $f$ is continuous.)

2. Work problem #12 on page 92 of the text.

3. Work problem #21 on page 94 of the text.

4. Let $(X, \mathscr{M}, \mu)$ be a measure space, and suppose that $1 \leq p < q < \infty$.

(a) Show that $L^p(X) \not\subset L^q(X)$ if and only if $X$ contains sets of arbitrarily small positive measure.

(b) Show that $L^q(X) \not\subset L^p(X)$ if and only if $X$ contains sets of arbitrarily large finite measure.

(c) What does this imply for the spaces $L^p(\mathbf{R})$ and $L^p([0, 1])$ (with respect to Lebesgue measure and $1 \leq p < \infty$)? How about the spaces $\ell^p$?

(Hints: this is essentially problem #5 on page 186 of the text. There are hints for the "if" portions of parts (a) and (b) there. For the "only if" directions, first note that if $f$ is bounded and in $L^p$, then $f$ is in $L^q$. Also, if $\mu(X) < \infty$, then $L^q(X) \subset L^p(X)$ by Hölder — see Proposition 6.12 in the text.)