# MATH 101: GRADUATE LINEAR ALGEBRA
# WORKSHEET, DAY #1

Let $F$ be a field. An <u>$F$-vector space</u> is an abelian group $V$ under $+$ with additive identity 0 equipped with a scalar multiplication

$$F \times V \to V$$

$$(a, x) \mapsto ax$$

compatible with $+$: for all $a, b \in F$ and $x, y \in V$, we have

$$a(x+y) = ax + ay, \quad (a+b)x = ax + bx, \quad (ab)x = a(bx), \quad \text{and} \quad 1x = x.$$

Let $V$ be a vector space over $F$. Let $v_1, \dots, v_n \in V$. A linear combination of $v_1, \dots, v_n$ is

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The set of all vectors $w$ which are linear combinations of $v_1, \dots, v_n$ forms a ____________ $W \subseteq V$, and we say that $W$ is ________________________ by $v_1, \dots, v_n$.

A linear relation among $v_1, \dots, v_n$ is a linear combination which is equal to zero, i.e.,

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The vectors $v_1, \dots, v_n$ are called ________________________ if there is no nonzero linear relation among the vectors, i.e., if $c_1v_1 + \dots + c_nv_n = 0$ then ________________________; otherwise $v_1, \dots, v_n$ are called ________________________. By convention, the empty set is considered to be ________________________, and the span of the empty set is ____________.

*Date*: Monday, 11 September 2017.

Two vectors $v_1, v_2$ have no nonzero linear relation if and only if either

________________________ or ______________________________.

An ordered set $B = \{v_1, \ldots, v_n\}$ of vectors that is linearly independent and spans $V$ is called a ________________________ of $V$; for example, for $F^n = \{(a_1, \ldots, a_n) : a_i \in F\}$ we may take

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**Lemma.** *The set $B$ is a basis for $V$ if and only if every $w \in V$ can be written uniquely as a*

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**Proposition.** *Let $L = \{v_1, \ldots, v_n\} \subseteq V$ be a linearly independent ordered set, and let $v \in V$. Then the ordered set $\{v_1, \ldots, v_n, v\}$ is linearly independent if and only if*

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**Proposition.** *For any finite set $S$ which spans $V$, there exists a subset $B \subseteq S$ which is a basis for $V$.*

*Proof.* Suppose that $S = \{v_1, \ldots, v_n\}$ and that $S$ is not linearly independent. Then

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**Lemma.** *Let $V$ be a vector space with a finite basis. Then any spanning set of $V$ contains a basis, and any* ______________________________ *set $L$ can be extended by adding elements of $V$ to get a basis.*

**Corollary.** *Suppose $V$ has a finite basis $B$ with $\#B = n \in \mathbb{Z}_{\geq 0}$. Then any set of linearly independent vectors has* ________________________ *elements, and any spanning set has* ________________________ *elements.*

*Proof.* Let $L$ be a linearly independent set of vectors. By the lemma,

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**Corollary.** *If $V$ has a finite basis then any two bases of $V$ have the same cardinality.*

*Proof.* __________________________________________________________________

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Let $V$ be a vector space. Then the dimension of $V$ is defined to be ________________________________________ and is denoted $\dim_F V$, and $V$ is said to be ________________________ over $F$.

If $F$ is a finite field with $\#F = q$, then a vector space of dimension $n$ over $F$ has ____________ elements.

A function $\phi\colon V \to W$ is an $F$-linear map or a ________________________________________ if ________________________________________________________________________.

Let $\phi\colon V \to W$ be $F$-linear. If there exists an $F$-linear inverse $\psi\colon V \to W$, then we say $\phi$ is an ________________________; it is actually enough to check that $\phi$ is ____________.

**Theorem.** *Let $V$ be a vector space of dimension $n$. Then $V \simeq F^n$. In particular, any two vector spaces of the same finite dimension are isomorphic.*

*Proof.* Let $v_1, \ldots, v_n$ be a basis for $V$. Define the map

$$\phi : F^n \to V$$

$$\phi(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n.$$

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**Theorem.** *Let $V$ be a finite-dimensional vector space over $F$ and let $W$ be a subspace of $V$. Then the quotient $V/W$ is a vector space with*

$$\dim(V/W) = \underline{\hspace{8cm}},$$

*Proof.* Since $V$ is finite-dimensional, so is $W$ because

________________________________________. Let $W$ have dimension $m$ and let $w_1, \ldots, w_m$ be a basis for $W$. We extend this basis to a basis $w_1, \ldots, w_m, v_{m+1}, \ldots, v_n$ of $V$. Then the projection map $V \to V/W$ maps each $w_i$ to ____________ and therefore has image spanned by $v_{m+1} + W, \ldots, v_n + W$; these vectors are linearly independent because

___________________________________________________. So

$\dim(V/W) = $ _________________________. □

**Corollary.** *Let $\phi : V \to W$ be a linear transformation. Then*

$$\dim V = \dim \ker \phi + \dim \operatorname{img} \phi.$$

We also say that $\ker \phi$ is the ________________________ of $\phi$ and $\dim \ker \phi$ is the

________________________. The dimension of $\operatorname{img} \phi = \phi(V)$ is called the

________________________.

**Corollary.** *Let $\phi : V \to W$ be a linear transformation of vector spaces of the same finite dimension $n$. Then the following are equivalent:*

(a) *$\phi$ is an isomorphism;*

(b) *$\phi$ is injective;*

(c) *$\phi$ is surjective.*

*Proof.* ____________________________________________________________

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