# MATH 101: GRADUATE LINEAR ALGEBRA
# WEEKLY HOMEWORK #6

**Problem W6.1**. Let $R$ be a commutative ring, let $S \subset R$ be a multiplicatively closed set containing 1, and let $S^{-1}R = R[S^{-1}]$ be the localization at $S$. Let $\phi\colon R \to S^{-1}R$ be the ring homomorphism $r \mapsto r/1$.

(a) Let $I \subseteq R$ be an ideal. Then $I$ is an $R$-module, so we have defined $S^{-1}I \subseteq S^{-1}R$, and
$$S^{-1}I = \{a/s : a \in I, s \in S\}.$$
Show that $S^{-1}I$ is an ideal in $S^{-1}R$. Show that $S^{-1}I = S^{-1}R$ if and only if $I \cap S \neq \emptyset$.

(b) Show that every ideal $I' \subseteq S^{-1}R$ is of the form $I' = S^{-1}I$ for an ideal $I \subseteq R$.

(c) Show that there is a bijection between the *prime* ideals of $S^{-1}R$ and the prime ideals of $R$ disjoint from $S$.

**Problem W6.2**.

(a) Let $R$ be a Euclidean domain with norm $N$. Let
$$m = \min(\{N(a) : a \in R, a \neq 0\}).$$
Show that every nonzero $a \in R$ with $N(a) = m$ is a unit in $R$. Deduce that a nonzero element of norm zero in $R$ is a unit; show by an example that the converse of this statement is false.

(b) Let $F$ be a field and let $R = F[[x]]$. Show that $R$ is Euclidean. What does part (a) tell you about $R^\times$? What are the irreducibles in $R$, up to associates?

**Problem W6.3**. Let $R$ be a domain and let $M$ be an $R$-module. Elements $x_1, \dots, x_n \in M$ are *$R$-linearly independent* if whenever $a_1x_1 + \cdots + a_nx_n = 0$ with $a_i \in R$, then $a_1 = \cdots = a_n = 0$.

The *rank* of $M$ is the maximal number of $R$-linearly independent elements of $M$.

(a) Suppose that $M$ has rank $n$ and that $x_1, \dots, x_n$ is any maximal set of $R$-linearly independent elements of $M$. Let $N = Rx_1 + \cdots + Rx_n$ be the $R$-submodule generated by $x_1, \dots, x_n$. Prove that $N$ is isomorphic to $R^n$ and that the quotient $M/N$ is a torsion $R$-module. *[Hint: Show that the map $R^n \to N$ which sends the $i$th standard basis vector to $x_i$ is an isomorphism of $R$-modules.]*

(b) Prove conversely that if $M$ contains a submodule $N$ that is free of rank $n$ (i.e., $N \cong R^n$) such that the quotient $M/N$ is a torsion $R$-module then $M$ has rank $n$. *[Hint: Let $y_1, \dots, y_{n+1}$ be any $n+1$ elements of $M$. Use the fact that $M/N$ is torsion to write $r_iy_i$ as a linear combination of a basis for $N$ for some nonzero elements $r_i$ of $R$. Use an argument like Proposition 12.1.3 to show that the $r_iy_i$, and hence also the $y_i$, are linearly dependent.]*

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*Date*: Assigned Friday, 3 November 2017; due Friday, 10 November 2017.

(c) Let $R = \mathbb{Z}[x]$ and let $M = (2, x)$ be the ideal generated by 2 and $x$, considered as a submodule of $R$. Show that $\{2, x\}$ is not a basis of $M$. Show that the rank of $M$ is 1 but that $M$ is not free of rank 1.

**Problem W6.4**.

(a) Let $N \leq \mathbb{Z}^2$ be the submodule generated by $(2, 4)$ and $(8, 10)$. Write $\mathbb{Z}^2/N$ as a product of cyclic groups.

(b) Let $R$ be a PID. Let $M \subseteq R^n$ be an $R$-submodule such that

$$\#(R^n/M) = [R^n : M] = p$$

where $p \in \mathbb{Z}$ is prime and $p$ is a nonzerodivisor in $R$. Show that $M$ is free of rank $n$ and there is a basis $x_1, \ldots, x_n$ of $R^n$ and $q \in R$ such that $M = Rx_1 \oplus \cdots \oplus Rqx_n$ and $[R : (q)] = p$.

**Problem W6.5**.

(a) Prove that two $2 \times 2$ matrices over $F$ which are not scalar matrices are similar if and only if they have the same characteristic polynomial.

(b) Prove that two $3 \times 3$ matrices are similar if and only if they have the same characteristic and minimal polynomials. Give an explicit counterexample to this assertion for $4 \times 4$ matrices.

**Problem W6.6**. Find all similarity classes of $6 \times 6$ matrices over $\mathbb{Q}$ with minimal polynomial $(x + 2)^2(x - 1)$. [It suffices to give all lists of invariant factors and write out some of their corresponding matrices.]