MATH 101: GRADUATE LINEAR ALGEBRA
WEEKLY HOMEWORK #4

If not otherwise specified, let R be a ring (with 1) and let M be a (left) R-module.

Problem W4.1. Let R be a (commutative integral) domain.

(a) Let k be a field and suppose that R is a finite-dimensional k-algebra. Show that R is a field.
(b) Show that if R is finite ($\#R < \infty$), then R is a field.

Problem W4.2. Let N be an R-submodule of M. Prove that if both M/N and N are finitely generated, then so is M.

Problem W4.3. M is called *irreducible* if $M \neq \{0\}$ and the only R-submodules of M are $\{0\}$ and M.

(a) Show that M is irreducible if and only if $M \neq \{0\}$ and M is a cyclic module with any nonzero element as generator.
(b) Determine all the irreducible $\mathbb{Z}$-modules.
(c) Suppose that R is commutative. Show that M is irreducible if and only if $M \simeq R/J$ where J is a maximal ideal of R.
(d) Show (again over an arbitrary ring R) that if M_1, M_2 are irreducible R-modules, then any R-module homomorphism $M_1 \to M_2$ is either zero or an isomorphism. Deduce that if M is irreducible then $\mathrm{End}_R(M)$ is a division ring.

Problem W4.4. Suppose R is commutative.

(a) Let M_i be R-modules for $i \in I$ and let N be another R-module. Show that

$$\mathrm{Hom}_R\left(\bigoplus_{i \in I} M_i, N\right) \simeq \prod_{i \in I} \mathrm{Hom}_R(M_i, N)$$

are isomorphic as R-modules.

(b) Let N_i be R-modules for $i \in I$. Show that there is an injective map

$$\bigoplus_{i \in I} \mathrm{Hom}_R(M, N_i) \hookrightarrow \mathrm{Hom}_R\left(M, \bigoplus_{i \in I} N_i\right)$$

of R-modules that is an isomorphism when either $\#I < \infty$ or when M is finitely generated.

[Hint: Make a big deal about why it is the direct product in one case and the direct sum in another. Of course there is only a distinction when $\#I = \infty$.]

Problem W4.5. Let $R = \mathbb{Z}[x]$ be the ring of polynomials in a variable x with integer coefficients. Let $I = (2, x)$ be the ideal generated by 2 and x in R.

(a) Show that $R/I \simeq \mathbb{Z}/2\mathbb{Z}$ as rings and as R-modules.
(b) Show that $2 \otimes 2 + x \otimes x \in I \otimes_R I$ is nonzero and cannot be written as a simple tensor (in the form $a \otimes b$ for some $a, b \in I$).
(c) The element $\alpha = 2 \otimes x - x \otimes 2 \in I \otimes_R I$ is nonzero. Compute its annihilator $\mathrm{ann}(\alpha) = \{r \in R : r\alpha = 0\}$ as an ideal in R. Show that the R-submodule $R\alpha \subseteq I \otimes_R I$ generated by α is isomorphic as an R-module to R/I.

Date: Assigned Monday, 16 October 2017; due Monday, 23 October 2017.