MATH 101: GRADUATE LINEAR ALGEBRA WEEKLY HOMEWORK #3

Problem W3.1. Let V, W be finite-dimensional inner product spaces.

(a) Let $\phi\colon V \to V$ be a self-adjoint linear operator. Recall that $\psi\colon V \to V$ is *positive semidefinite* if $\langle \psi(x), x\rangle \geq 0$ whenever $x \neq 0$. Show that ϕ is positive semidefinite if and only if all of the eigenvalues of ϕ are nonnegative.

(b) Now let $\phi\colon V \to W$ be linear. Show that $\phi^*\phi$ and $\phi\phi^*$ are positive semidefinite.

(c) Show that $\operatorname{rk}(\phi^*\phi) = \operatorname{rk}(\phi\phi^*) = \operatorname{rk}(\phi)$.

Problem W3.2. Let $V = \mathbb{R}^n$ be the standard inner product space. Let

$$S = \{x \in V : \|x\|^2 = 1\}$$

be the $(n-1)$-dimensional sphere in V.

(a) Suppose that $x, y \in S$ have $\langle x, y\rangle = 0$. Show that $\cos(t)x + \sin(t)y$ lies on S for all $t \in \mathbb{R}$.

(b) Let $\phi : V \to V$ be a self-adjoint linear map. By vector calculus, the function $x \mapsto \langle x, \phi(x)\rangle$ achieves a maximum at some point $p \in S$: briefly explain why. Let $y \in S$ satisfy $\langle p, y\rangle = 0$. Consider the function

$$f(t) = \langle \cos(t)p + \sin(t)y, \phi(\cos(t)p + \sin(t)y)\rangle.$$

Show that $\langle p, \phi(y)\rangle = 0$.

(c) Let $W = \operatorname{span}(\{p\})$. Show that $W^\perp$ is ϕ-invariant and then conclude that W is ϕ-invariant. Conclude that p is an eigenvector!

(d) Parlay the argument of (c) into an inductive proof that V has an orthonormal basis of vectors that are eigenvectors for ϕ.

[Note: This argument inductively gives a different "physical" or "geometric" proof that ϕ has an orthonormal basis of eigenvectors: we find an eigenvector by maximizing ϕ on the sphere!]

Problem W3.3. In each part, let $\phi\colon V \to V$ be the projection on the subspace W_1 along the subspace W_2, where $V = W_1 \oplus W_2$.

(a) Show that ϕ is an orthogonal projection (i.e., $W_2 = W_1^\perp$) if and only if $\|\phi(x)\| \leq \|x\|$ for all $x \in V$. *[Hint: Let $w_1 \in W_1$ and $w_2 \in W_2$ be nonzero and $c \in \mathbb{R}$, and let $w = cw_1 + w_2$. Show that*

$$2c\,\mathrm{Re}\langle w_1, w_2\rangle + \|w_2\|^2 \geq 0.$$

If $\langle w_1, w_2\rangle \neq 0$, derive a contradiction by a choice of c; conclude that $\langle w_1, w_2\rangle = 0$ for all w_1, w_2.]

(b) What can you conclude if ϕ is unitary? [So don't confuse a projection that is orthogonal with an orthogonal projection!]

(c) Suppose that ϕ is *normal* over $F = \mathbb{C}$. Prove that ϕ is an orthogonal projection.

Date: Assigned Wednesday, 4 October 2017; due Wednesday, 11 October 2017.

Problem W3.4. Let $\phi, \psi \colon V \to V$ be normal operators on a finite-dimensional complex inner product space V. Suppose that $\phi\psi = \psi\phi$. Prove that there exists an orthonormal basis for V consisting of (simultaneous) eigenvectors for ϕ and ψ.