# MATH 101: GRADUATE LINEAR ALGEBRA
# WEEKLY HOMEWORK #2

**Problem W2.1**. Let $\phi\colon V \to W$ be an $F$-linear map, and let $\phi^*\colon W^* \to V^*$ be the dual map, defined via pullback. Show that

$$\operatorname{img} \phi^* = \operatorname{ann}(\ker \phi).$$

**Problem W2.2**. Let $V$ be a finite-dimensional vector space over a field $F$, and let $W_1, W_2$ be subspaces.

(a) Prove that $W_1 = W_2$ if and only if $\operatorname{ann}(W_1) = \operatorname{ann}(W_2)$.
(b) Show $\operatorname{ann}(W_1 + W_2) = \operatorname{ann}(W_1) \cap \operatorname{ann}(W_2)$ and $\operatorname{ann}(W_1 \cap W_2) = \operatorname{ann}(W_1) + \operatorname{ann}(W_2)$.

For a heightened sense of self-satisfaction, you could make it clear in your argument where you actually use that $V$ is finite-dimensional. Which of the statements are still true when $V$ is infinite-dimensional?

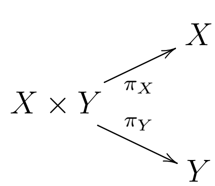
**Problem W2.3**. Let $V, W$ be $F$-vector spaces, let $v_1, \dots, v_n \in V$ be linearly independent, and let $w_1, \dots, w_n \in W$ be arbitrary. Suppose that

$$\sum_{i=1}^{n} v_i \otimes w_i = 0 \in V \otimes_F W.$$

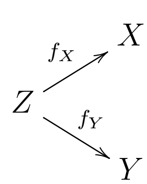
Show that $w_i = 0$ for all $i = 1, \dots, n$. Conclude that $v \in V$ and $w \in W$ have $v \otimes w = 0$ if and only if $v = 0$ or $w = 0$.

**Problem W2.4**. In class, we showed that the tensor product is characterized by a universal property. Perhaps the simplest situation of a universal property is the following.

Let $X, Y$ be sets. The cartesian product $X \times Y$ has its two projection maps:

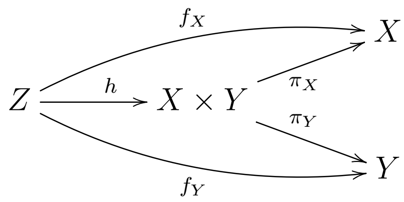



Show that the product $X \times Y$ is *universal* in this respect: for every set $Z$ and maps




*Date*: Assigned Friday, 22 September 2017; due Monday, 2 October 2017.

of sets, there exists a unique map $h\colon Z \to X \times Y$ such that the diagram




commutes.

**Problem W2.5**. Let $F$ be a field, let $V$ be a finite-dimensional $F$-vector space, and let $T : V \times V \to F$ be a nondegenerate symmetric bilinear form. Let $W \subseteq V$ be a subspace.

Define

$$W^\perp = \{v \in V : T(v, W) = 0\} = \{v \in V : T(v, w) = 0 \text{ for all } w \in W\}.$$

(a) Show that the map

$$\begin{aligned} V &\to V^* \\ v &\mapsto T_v = T(v, -) \end{aligned}$$

maps $W^\perp$ isomorphically to $\operatorname{ann}(W)$.

(b) Deduce that $\dim V = \dim W + \dim W^\perp$.

(c) Suppose that $T|_{W\times W}$ is nondegenerate (accordingly, we say that $W$ is a *nondegenerate* subspace under $T$). Show that $V = W \oplus W^\perp$. In this case, we say $W^\perp$ is the *orthogonal complement* of the nondegenerate subspace $W$.

(d) Define the *orthogonal projection* onto $W$ (as a linear operator on $V$). Let $V = \mathbb{R}^3$ have the standard inner product and let

$$W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}.$$

Compute the matrix of the orthogonal projection onto $W$ with respect to the standard basis.