MATH 101: GRADUATE LINEAR ALGEBRA
DAILY HOMEWORK #21

Problem 21.1. Let R be a commutative ring such that $\mathfrak{a} = R \smallsetminus R^\times$ is an ideal. Show that $\mathfrak{a}$ is the unique maximal ideal in R.

Problem 21.2. Let $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$ be the subring generated by 1 and $\sqrt{-5}$. Let $\mathfrak{a} = (2, 1 + \sqrt{-5}) \subseteq R$, and let $\mathfrak{p} \subseteq R$ be a prime ideal.

(a) Show that $\mathfrak{a}$ is prime.
(b) Show that $\mathfrak{p} \cap \mathbb{Z}$ is a nonzero prime ideal.
(c) If $2 \notin \mathfrak{p}$, show that $\mathfrak{a}_{(\mathfrak{p})} = R_{(\mathfrak{p})}$.
(d) If $2 \in \mathfrak{p}$, show that $\mathfrak{p} = \mathfrak{a}$ and that $\mathfrak{a}_{(\mathfrak{p})} = (1 + \sqrt{-5})R_{(\mathfrak{p})}$.
(e) Conclude that $\mathfrak{a}_{(\mathfrak{p})}$ is locally free.

Date: Assigned Friday, 27 October 2017; due Monday, 30 October 2017.