MATH 101: ALGEBRA I
WORKSHEET, DAY #4

Fill in the blanks.

Let F be a field. An F-module V is also known as a ___F-vector space___ over F, and an F-module homomorphism $\phi : V \to W$ is called a ________________________.

Let V be a vector space over F. Let $v_1, \dots, v_n \in V$. A *linear combination* of $v_1, \dots, v_n$ is

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The set of all vectors w which are linear combinations of $v_1, \dots, v_n$ forms a ____________ $W \subset V$, and we say that W is ________________________ by $v_1, \dots, v_n$.

A *linear relation* among vectors $v_1, \dots, v_n$ is a linear combination which is equal to zero, i.e.,

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The vectors $v_1, \dots, v_n$ are called ________________________ if there is no nonzero linear relation among the vectors, i.e., if $c_1v_1 + \cdots + c_nv_n = 0$ then ________________________; otherwise $v_1, \dots, v_n$ are called ________________________. By convention, the empty set is considered to be ________________________, and the span of the empty set is ____________. Two vectors v_1, v_2 have no nonzero linear relation if and only if either

________________________ or ______________________________.

An ordered set $B = \{v_1, \dots, v_n\}$ of vectors that is linearly independent and spans V is called a ________________________ of V; for example, for $F^n = \{(a_1, \dots, a_n) : a_i \in F\}$ we

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may take

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Lemma. *The set B is a basis for V if and only if every $w \in V$ can be written uniquely as a*

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Proposition. *Let $L = \{v_1, \ldots, v_n\} \subset V$ be a linearly independent ordered set, and let $v \in V$. Then the ordered set $\{v_1, \ldots, v_n, v\}$ is linearly independent if and only if*

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Proposition. *For any finite set S which spans V, there exists a subset $B \subset S$ which is a basis for V.*

Proof. Suppose that $S = \{v_1, \ldots, v_n\}$ and that S is not linearly independent. Then

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Lemma. *Let V be a vector space with a finite basis. Then any spanning set of V contains a basis, and any ________________ set L can be extended by adding elements of V to get a basis.*

Corollary. *Suppose V has a finite basis B with $\#B = n$. Then any set of linearly independent vectors has at most ________ elements, and any spanning set has ____________ elements.*

Proof. Let L be a linearly independent set of vectors. By the lemma,

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Corollary. *If V has a finite basis then any two bases of V have the same cardinality.*

Proof. ____

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Let V be a vector space with a finite basis. Then the dimension of V is defined to be ____________________________ and is denoted $\dim_F V$, and V is said to be ________________ over F.

If F is a finite field with $\#F = q$, then a vector space of dimension n over F has ________ elements.

Theorem. *Let V be a vector space of dimension n. Then $V \simeq F^n$. In particular, any two vector spaces of the same finite dimension are isomorphic.*

Proof. Let $v_1, \ldots, v_n$ be a basis for V. Define the map

$$\phi : F^n \to V$$

$$\phi(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n.$$

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Theorem. *Let V be a finite-dimensional vector space over F and let W be a subspace of V. Then the quotient V/W is a vector space with*

$$\dim(V/W) = \underline{\hspace{8cm}},$$

Proof. Since V is finite-dimensional, so is W because ____________________________________. Let W have dimension m and let $w_1, \ldots, w_m$ be a basis for W. We extend this basis to a basis $w_1, \ldots, w_m, v_{m+1}, \ldots, v_n$ of V. Then the projection map $V \to V/W$ maps each w_i to __________ and therefore has image spanned by $v_{m+1} + W, \ldots, v_n + W$; these vectors are linearly independent because ____. So $\dim(V/W) =$ ____________________. □

Corollary. *Let $\phi : V \to W$ be a linear transformation. Then*

$$\dim V = \dim \ker \phi + \dim \operatorname{img} \phi.$$

We also say that $\ker \phi$ is the ____________________of ϕ and $\dim \ker \phi$ is the ____________________. The dimension of $\operatorname{img} \phi = \phi(V)$ is called the ____________________.

Corollary. *Let $\phi : V \to W$ be a linear transformation of vector spaces of the same finite dimension n. Then the following are equivalent:*

(a) ϕ *is an isomorphism;*

(b) ϕ *is injective;*

(c) ϕ *is surjective.*

Proof. ____

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