MATH 101: ALGEBRA I
MIDTERM EXAM SOLUTIONS

Problem 1. Let G be a group. Indicate if the following statements are true or false. If true, give a proof; if false, give an explicit counterexample.

(a) If $H, H' \trianglelefteq G$ and $G/H \simeq G/H'$, then $H \simeq H'$.
(b) If $H, H' \trianglelefteq G$ and $H \simeq H'$, then $G/H \simeq G/H'$.
(c) If K, K' are groups and $G \times K \simeq G \times K'$, then $K \simeq K'$.

Solution. Part (a) is false. Take $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $H = \langle (0,1) \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ and $H' = \langle (2,0) \rangle \simeq \mathbb{Z}/2\mathbb{Z}$. Then $G/H \simeq \mathbb{Z}/4\mathbb{Z}$ and $G/H' \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Part (b) is also false. Take $G = \mathbb{Z}$ and $H = \mathbb{Z}$ and $H' = 2\mathbb{Z} \simeq \mathbb{Z}$; then $G/H = \{1\} \not\simeq \mathbb{Z}/2\mathbb{Z} \simeq G/H'$.

Part (c) is also also false. Take G to be a countable product of copies of $\mathbb{Z}/2\mathbb{Z}$, and $K = \mathbb{Z}/2\mathbb{Z}$ and $K' = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then $K \not\simeq K'$, but $G \simeq G \times K \simeq G \times K'$ are all isomorphic. (It turns out that if G, K, K' are finite, then the result becomes true, but this is not easy to prove.)

Problem 2. Let R be a Euclidean domain with norm N.

(a) Let
$$m = \min(\{N(a) : a \in R, a \neq 0\}).$$
Show that every nonzero $a \in R$ with $N(a) = m$ is a unit in R.
(b) Deduce that a nonzero element of norm zero in R is a unit; show by an example that the converse of this statement is false.
(c) Let F be a field and let $R = F[[x]]$. Show that R is Euclidean. What does part (a) tell you about $R^\times$? What are the irreducibles in R, up to associates?

Solution. Let a be a nonzero element of norm m. Then we can write $1 = qa + r$ with $r = 0$ or $N(r) < N(a) = m$. We cannot have the latter, since m is the smallest such, hence $r = 0$ so $1 = qa$ and hence $a \in R^*$, which proves (a). For (b), if there is an element of norm zero then $m = 0$ so by (a) every nonzero element of norm zero is a unit. The converse of this statement is false, namely, that every unit has norm zero: the ring $\mathbb{Z}[i]$ is a Euclidean domain and $a \in \mathbb{Z}[i]$ is a unit with respect to the complex norm if and only it has norm 1.

Finally, part (c). For $\alpha = a_n x^n + \cdots \in F[[x]]$ with $a_n \neq 0$, we define the norm $N(\alpha) = n \geq 0$. Then R is Euclidean under this norm as follows. Let $\alpha, \beta \in R$ with $\beta \neq 0$. If $N(\alpha) < N(\beta)$, then we can write $\alpha = 0\beta + \alpha$. Otherwise $N(\alpha) \geq N(\beta)$, and we claim $\beta \mid \alpha$, i.e., $\alpha = (\alpha/\beta)\alpha + 0$ with $\alpha/\beta \in F[[x]]$. Indeed, write $\alpha = x^{N(\alpha)}\alpha_0(x)$ and $\beta = x^{N(\beta)}\beta_0(x)$ with $\beta_0(x) = b_0 + \dots$ and $b_0 \neq 0$; we showed in class that $\beta_0(x) \in F[[x]]^\times$ by solving linear equations, so $\alpha/\beta = x^{n-m}\alpha_0(x)\beta_0(x)^{-1} \in F[[x]]$. Therefore $F[[x]]$ is Euclidean under this norm. Then part (a) reminds us that $R^\times = F[[x]]^\times$ consists of the elements with nonzero constant term, reading off the definition of the norm. The only irreducible, up to associates, is x. Indeed, we know that $F[[x]]$ is a UFD so irreducibles are the same as primes, and $F[[x]]/(x) \simeq F$ so x is irreducible; and any $\alpha(x) = x^{N(\alpha)}\alpha_0(x) \neq 0$ with $\alpha_0(x) \in F[[x]]^\times$ is then a factorization of $\alpha(x)$ as a power of the irreducible x (times a unit).

Problem 3. Let F be a field and let $V = \mathrm{Mat}_{2\times 3}(F)$ be the F-vector space of 2×3-matrices.

(a) The group $\mathrm{GL}_2(F)$ acts on V by left multiplication. For $M, M' \in V$, the relation $M \sim M'$ if and only if $M' = AM$ for some $A \in \mathrm{GL}_2(F)$ defines an equivalence relation on V.
What are the equivalence classes (i.e., the *orbits* of the action)?
(b) Show that this action $\mathrm{GL}_2(F) \circlearrowright V$ induces an injective group homomorphism
$$\phi : \mathrm{GL}_2(F) \hookrightarrow \mathrm{Aut}_F(V).$$

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(c) Under the isomorphism $\mathrm{Aut}_F(V) \simeq \mathrm{GL}_6(F)$ given by the basis of matrix units, describe ϕ explicitly.

Solution. For (a), a matrix $A \in \mathrm{GL}_2(F)$ acts on the left by row operations. So every $M \in V$ can be put into reduced row echelon form by this action. By linear algebra, the reduced row echelon form is unique. The possible forms (choosing pivots) are

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $*$ denotes an arbitrary element of F.

For (b), we get a map $\phi : \mathrm{GL}_2(F) \to \mathrm{End}_F(V)$ because matrix multiplication map is F-linear:

$$A(M + cM') = AM + cAM' \quad \text{for all } A \in \mathrm{GL}_2(F),\ M, M' \in V, \text{ and } c \in F.$$

The map is a homomorphism because this holds for matrix multiplication:

$$\phi(AB)(M) = (AB)M = A(B(M)) = (\phi(A) \circ \phi(B))(M)$$

for all $A, B \in \mathrm{GL}_2(F)$ and $M \in V$. In a group action, we always have the image landing in the symmetric group on the set (acting bijectively), and indeed the inverse to A is A^{-1}, so the image lands in $\mathrm{Aut}_F(V)$. Finally, the map is injective: take $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ to see that $AM = M$ implies $A = 1$.

For (c), taking the basis $e_{11}, e_{21}, e_{12}, e_{22}, e_{13}, e_{23}$, and noting that matrix multiplication acts independently on column vectors, we compute that $\phi(A)$ is the block diagonal matrix with three copies of A down the diagonal, for each $A \in \mathrm{GL}_2(F)$.

Problem 4. For the purposes of this exercise, we say that an isomorphism of F-vector spaces is *natural* if it does not depend on a choice of basis.

Let F be a field and let V, W be finite-dimensional vector spaces over F. Show that there is a (well-defined) natural isomorphism of F-vector spaces

$$\phi : V^* \otimes_F W \xrightarrow{\sim} \mathrm{Hom}_F(V, W).$$

Solution. To start, note that

$$\dim_F(V \otimes_F W^*) = \dim_F(V) \dim_F(W^*) = \dim_F(V) \dim_F(W) = \dim_F \mathrm{Hom}_F(V, W)$$

so there certainly is an isomorphism. By this dimension count, it is enough to exhibit a natural injective F-linear map.

There is really only one thing we could write down: given a simple tensor $f \otimes w \in V^* \otimes W$, we define $\phi(f \otimes w) \in \mathrm{Hom}_F(V, W)$ by $\phi(f \otimes w)(v) = f(v)w$, and we extend the map to a sum of simple tensors by linearity. The map $\phi(f \otimes w)$ is indeed F-linear, since

$$\phi(f \otimes w)(v + cv') = f(v + cv')w = f(v)w + cf(v')w = \phi(f \otimes w)(v) + c\phi(f \otimes w)(v').$$

To show tjat ϕ is well-defined, we observe first that

$$\phi((f + cf') \otimes w)(v) = (f + cf')(v)w = f(v)w + cf'(v)w = \phi(f \otimes w)(v) + c\phi(f' \otimes w)(v)$$

for all $f, f' \in V^*$, $c \in F$, $v \in V$, and $w \in W$, so we conclude that

$$\phi((f + cf') \otimes w) = \phi(f \otimes w) + c\phi(f' \otimes w).$$

In a similar fashion, one can show that

$$\phi(f \otimes (w + cw')) = \phi(f \otimes w) + c\phi(f \otimes w').$$

and immediately we see that the map ϕ is F-linear.

To show that ϕ is injective, we may choose a basis $v_1, \dots, v_n$ of V and $w_1, \dots, w_m$ of W. Let v_i^* be the dual basis of V^*. Then $v_i^* \otimes w_j$ is an F-basis of $V^* \otimes W$. Let $\sum_{i,j} c_{ij} v_i^* \otimes w_j \in \ker \phi$. Then for all $v \in V$, we have

$$\phi\Bigg(\sum_{i,j} c_{ij} v_i^* \otimes w_j\Bigg)(v) = \sum_{i,j} c_{ij} v_i^*(v) w_j = \sum_j \Bigg(\sum_i c_{ij} v_i^*(v)\Bigg) w_j = 0.$$

Since the w_j are linearly independent, we have $\sum_i c_{ij} v_i^*(v) = 0$ for all j and all $v \in V$, which means $\sum_i c_{ij} v_i^* = 0$; but the v_i^* are linearly independent, so $c_{ij} = 0$ for all i, j.