ERRATA:
ON THE ARITHMETIC DIMENSION OF TRIANGLE GROUPS

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This note gives errata for the article *On the arithmetic dimension of triangle groups* [1]. The authors thank Giovanni Panti and Cherng-tiao Perng.

(1) (1.2): there is a $\mathbb{Q}$ missing, so it should read
$$E = \mathbb{Q}(\operatorname{Tr}\Delta^{(2)}) = \mathbb{Q}(\{\operatorname{Tr}(\delta^2) : \delta \in \Delta\}) = \mathbb{Q}(\lambda_a, \lambda_b, \lambda_c, \lambda_{2a}\lambda_{2b}\lambda_{2c})$$
(2) (1.3): in two lines before, this is the preimage of $\Delta^{(2)}$, and the isomorphism requires $b \neq \infty$. (When $b = \infty$, necessarily $A \simeq \mathrm{M}_2(E)$ because δ_b is parabolic, so A has a zerodivisor.)

(3) Lemma 1.9: the cardinality signs are missing, so it should read
$$\begin{aligned}\operatorname{adim}(a,b,c) &= \#\{k \in (\mathbb{Z}/2m\mathbb{Z})^\times/H : \sigma_k(\beta) > 0\}\\ &= \#\{k \in (\mathbb{Z}/2m\mathbb{Z})^\times/H : \kappa(a,b,c;k) < 0\}.\end{aligned}$$
(4) Proof of Theorem 2.1, "after permuting a, b, c and choosing ϵ, we see that if $t = 1$": This also covers the (apparently unexplained) case $t = 0$.

(5) Theorem 2.1: in case (b), if one of a, b, c is equal to 2, then the argument for H_1 is not correct: the congruence $k \equiv \pm 1 \pmod 2$ does not give distinct signs. This issue arises also in the characterization of $H = H_1$: since $\cos(\pi/2) = 0$ the condition
$$\cos\frac{k\pi}{a}\cos\frac{k\pi}{b}\cos\frac{k\pi}{c} = \cos\frac{\pi}{a}\cos\frac{\pi}{b}\cos\frac{\pi}{c}$$
is vacuous. The corrected statement: if $2 \in \{a, b, c\}$, then $\#H_1 = \max(2, 2^u) \geq 4$ and $H = H_1$; otherwise, if $a, b, c \geq 3$, the original conclusion holds.

(6) Theorem 2.12: the statement in (c), and its proof, concern $H = H_1$ (not $H = H_2$).

(7) Lemma 3.4: there is a typo in the inequality, and the statement should read: $\kappa(a, b, c; k) \geq 0$ if and only if (3.5) holds. In the proof, we let $f(z) = \kappa(a, b, c; k)$ and considered when $f(z) \geq 0$. The remaining statements using Lemma 3.4 (Corollary 3.6, Proposition 4.6, Theorem 5.2) use the corrected Lemma 3.4; before (3.8) the inequality should be switched to > 0; the rest of the paper remains unchanged.

(8) Proof of Lemma 3.4, "The discriminant simplifies as": This should read
$$\sqrt{t^2 - 4n} = \sqrt{4\cos^2\frac{k_a\pi}{a}\cos^2\frac{k_b\pi}{b} - 4\cos^2\frac{k_a\pi}{a} - 4\cos^2\frac{k_b\pi}{b} + 4}$$
(so the final -4 should be $+4$), as shown on the next line.

(9) Proof of Proposition 4.6, "In particular $1 \leq 3q/a$": Should be "In particular, $1 < 3q/a$."

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(10) Beginning of section 5: should be "section", not "chapter".
(11) Line 2 of Algorithm 1: Typo, should be $\max(48, 2\mathtt{r})$ (according to Lemma 4.10).
(12) Line 5 of Algorithm 3: "for" should be **for**, and it is missing **do**. This step just initializes `divisors` to be an array of 1s.
(13) Line 5 of Algorithm 4: should be ($\mathtt{a}$,$\mathtt{b}$,$\mathtt{c}$,r).
(14) Theorem 5.2: To be more precise, this theorem proves the correctness of FIND_R_ARITHMETIC.
(15) Proof of Theorem 5.2: The middle paragraph is a bit muddled. Here is a revised version that hopefully makes the logic clear.

We have from Lemma 3.4 that for every (a, b, c) and every $k \in \mathbb{Z}_{>0}$ with $k \in (\mathbb{Z}/2m\mathbb{Z})^\times$, if $\kappa(a, b, c; k) > 0$ then

$$c < \frac{k_c}{|k_a/a + k_b/b - 1|} \leq \frac{k}{|k_a/a + k_b/b - 1|} = \frac{kab}{|k_a b + k_b a - ab|}.$$

Suppose (a, b, c) is r-arithmetic, and consider the set of primes $q < c/2$ with $q \nmid m$. As in the proof of Lemma 4.3, these primes are distinct in $(\mathbb{Z}/2m\mathbb{Z})^\times/H$, so there at most $r - 1$ of them for which $\kappa(a, b, c; q) < 0$; so among any r of them, there is at least one with $\kappa(a, b, c; q) > 0$. Putting these observations together, we find that in any set of r primes with $q \nmid ab$, there is a prime q in the set where at least one of the following holds:

$$\text{either } q \mid c \text{ or } c < 2q \text{ or } c < \left\lceil \frac{qab}{|q_a b + q_b a - ab|} \right\rceil.$$

To finish, suppose (a, b, c) is r-arithmetic, $c > 2{*}\mathtt{maxNDP}$, and $c > \mathtt{bound}$, where `bound` is in the keyset of `boundToPrimes`. Let

$$B = \{q : q \in \mathtt{boundToPrimes}[\mathtt{bound}'] \text{ and } \mathtt{bound}' \leq \mathtt{bound}\}.$$

Then there exist at most $r - 1$ primes $q \in B$ that do not divide c. The algorithm partitions B into r sets, and lets each `divisor` in `divisors` be the product of primes in one such set. Hence, c must be a multiple of at least one `divisor` in `divisors`. Therefore, the algorithm checks all possible r-arithmetic triples (a, b, c).

References

[1] Steve Nugent and John Voight, *On the arithmetic dimension of triangle groups*, Math. Comp. **86** (2017), no. 306, 1979–2004.