ERRATA: *CHARACTERIZING QUATERNION RINGS OVER AN ARBITRARY BASE*

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This note gives errata and addenda for the article *Characterizing quaternion rings over an arbitrary base* [2].

Errata

(1) We use the term *isometry* when the term *similarity* is probably better for our definition of morphisms of quadratic forms. A similarity between two free quadratic modules is required to have invertible determinant—rather than an isometry, which is presumed to have determinant 1. It would have been clearer to use the word *similarity* in our context.
(2) Balmer–Calmès [1] have also studied a notion similar to our notion of a parity factorization (over schemes over $\mathbb{Z}[1/2]$): they call it a *quadratic alignment*.
(3) In Lemma 1.5, the hypothesis $\mathrm{rk}(B) > 2$ (not yet imposed) was omitted.
(4) Before Lemma 2.1, B/R is claimed to be a (faithfully) projective R-module. For this, a lemma is used which assumes that R is noetherian, but this is not used in the proof; the result is correct as stated by noetherian reduction.
(5) §3 "An equivalence of categories", and end of the proof of Theorem 3.13: it is claimed that an equivalence of categories has been proven. It can indeed be shown, but it is not part of the theorem.
(6) Example 3.4: should be $q_{\max}(x, y, z) = x^2 + y^2 + z^2 + xy + xz$.
(7) Page 128, paragraph 2: The reference "Knus [11, §III.3]" should be "Knus [13, §III.3]".
(8) Proof of Proposition 4.10: the last equalities should read

$$\chi_L(\xi; T) = \mu(\xi; T)(\mu(\xi; T) - (u'x + v'y + w'z)T)$$
$$\chi_R(\xi; T) = \mu(\xi; T)(\mu(\xi; T) + (u'x + v'y + w'z)T).$$

(The algebra trace is R-linear, so there must be the coefficients x, y, z.)

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To be totally explicit here, the R-algebra homomorphism $B \to \mathrm{M}_4(R)$ given by left-multiplication on the basis $1, i, j, k$ is given by

$$\begin{aligned} B &\to \mathrm{M}_4(R) \\ i &\mapsto \begin{pmatrix} 0 & -bc & cw & u'w' - uw \\ 1 & u & 0 & w - w' \\ 0 & 0 & u' & b \\ 0 & 0 & -c & u \end{pmatrix} \\ j &\mapsto \begin{pmatrix} 0 & u'v' - uv & -ac & au \\ 0 & v & 0 & -a \\ 1 & u - u' & v & 0 \\ 0 & c & 0 & v' \end{pmatrix} \\ k &\mapsto \begin{pmatrix} 0 & bv & v'w' - vw & -ab \\ 0 & w' & a & 0 \\ 0 & -b & w & 0 \\ 1 & 0 & v - v' & w \end{pmatrix}. \end{aligned}$$

so e.g.

$$\mathrm{Tr}_L(xi + yj + zk) = 2(ux + vy + wz) + (u'x + v'y + w'z).$$

(9) Page 132, last paragraph: In the definition of crossed product, it is hazardous to use b for the bilinear form since it also is used as a coefficient in the multiplication table (Q). Also, the definition of crossed product is incorrect. Instead of assuming that J is an invertible S-module, one should instead only assume that J is projective of rank 2 as an R-module and is a *traceable* S-module [3, p. 1758], so that the trace maps $\mathrm{Tr} : S \to R$ induced by multiplication on S and on J are equal. (Further comments and remarks are made in the revised version.) Finally, the expression $xu = \overline{h}x$ should read $xu = \overline{u}x$.

Addenda

(1) Bjorn Poonen pointed out that the parity factorization can be more simply described. We are solving the equation $P^{\otimes 2} \otimes Q \simeq R$ with R fixed. In this equation, it is enough to remember P, since there is then a unique solution Q.

The notion of parity factorization can also be replaced by remembering more simply a class in $\mathrm{Pic}(R)$ (suggested by Bjorn Poonen). That is to say, there is a bijection

$$\left\{ \begin{array}{c} \text{Isomorphism classes of quaternion} \\ \text{rings } B \text{ over } R \text{ equipped with a parity} \\ \text{factorization } p : P^{\otimes 2} \otimes Q \xrightarrow{\sim} \bigwedge^4 B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{quaternion rings } B \text{ over } R \\ \text{equipped with } [P] \in \mathrm{Pic}(R) \end{array} \right\}$$

which is functorial with respect to the base ring R: to a quaternion ring B equipped with a parity factorization $p : P^{\otimes 2} \otimes Q \xrightarrow{\sim} \bigwedge^4 B$, we associate the class $[P] \in \mathrm{Pic}(R)$, and conversely given B with $[P]$ we take the parity factorization

$$p : P^{\otimes 2} \otimes (\textstyle\bigwedge^4 B \otimes (P^\vee)^{\otimes 2}) \xrightarrow{\sim} \bigwedge^4 B.$$

References

[1] Paul Balmer and Baptiste Calmès, *Bases of total Witt groups and lax-similitude*, J. Algebra Appl. **11** (2012), no. 3, 1250045, 24 pp.

[2] John Voight, *Characterizing quaternion rings over an arbitrary base*, J. Reine Angew. Math. **657** (2011), 113–134.

[3] Melanie Wood, *Gauss composition over an arbitrary base*, Adv. Math. **226** (2011), no. 2, 1756–1771.