ORAL QUALIFYING EXAM QUESTIONS

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Below are some questions that I have asked on oral qualifying exams (starting in fall 2015).

1. Algebra

1.1. Core questions.

(1) Let R be a noetherian (commutative) domain, let M be a finitely generated R-module, and suppose $f : M \to M$ is a surjective R-module homomorphism. Show that f is injective—and then show that this conclusion fails if M is not finitely generated or if R is not noetherian.

(2) Consider the category of groups (under homomorphisms) and the category of sets (under maps). Define a map which associates to a group G the set $\{g \in G : g^2 = 1\}$. Show that this map is a functor. On the other hand, if we associate to G the set of elements of order 2, show that it is *not* a functor.

(3) Explain the proof that every PID is a UFD (in particular, why prime and irreducible are the same) and exhibit a UFD that is not a PID.

(4) Let S be the p-Sylow subgroup of $G = \mathrm{GL}_n(\mathbb{F}_q)$, where $q = p^a$. Compute $\#S$, decide when S is normal in G. Then show that there is a nonzero fixed point for the natural action of S on $\mathbb{F}_q^n$.

(5) Let ℓ be an odd prime. Describe the Sylow ℓ-subgroups of $G = \mathrm{SL}_2(\mathbb{F}_q)$ with $q = p^a$.

(6) Let R be a commutative ring and let S be a multiplicatively closed subset containing 1. Define the localization $R[S^{-1}]$, define a natural map $i_R : R \to R[S^{-1}]$, and explain the relationship between the prime ideals of R and the prime ideals of $R[S^{-1}]$. Give an example where i_R is not injective, and show that the association from R to $R[S^{-1}]$ is functorial in R (suitably interpreted).

(7) Let V be an $\mathbb{R}$-vector space with $\dim_{\mathbb{R}} V = 3$. Let $\phi\colon V \to V$ be an $\mathbb{R}$-linear operator. Show that V has a one-dimensional and a two-dimensional ϕ-invariant subspace.

(8) Let $R = \mathbb{Z}/4\mathbb{Z}$.
   (a) Give an example of an R-module that is not projective.
   (b) What are the simple R-modules? Give an example of an R-module that is not semisimple.
   (c) What are the possible localizations of R?

1.2. Representation theory of finite groups.

(1) Construct explicitly the irreducible 2-dimensional representation of S_3 by considering the rigid motions of an equilateral triangle in $\mathbb{R}^2$.

(2) Construct the character table of the quaternion group Q_8 of order 8.

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(3) Let $K \supseteq F$ be a finite Galois extension of fields. Then $G = \mathrm{Gal}(K \mid F)$ acts on K, linearly as an F-vector space, so K has the structure of an $F[G]$-module. Characterize K as an $F[G]$-module up to isomorphism.
(4) Let G be a group of order 21.
   (a) Show that either G is abelian or $G \simeq \mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ is a semidirect product.
   (b) What are the 1-dimensional representations of G?
   (c) Write down as much of the character table as you can. What size is the table? What entries do you know?

1.3. **Integral extensions.**

(1) Let A be a (commutative) domain. Define what it means for a ring B to be integral over A and give several equivalent formulations.
(2) Give a definition of Dedekind domain and several equivalent formulations. Is there an uncountable Dedekind domain?
(3) Let $L \supseteq K$ be a finite extension of fields and consider the trace pairing

$$\begin{aligned} L \times L &\to K \\ x, y &\mapsto \mathrm{Tr}_{L|K}(xy). \end{aligned}$$

Under what hypotheses is this pairing nondegenerate? What does the pairing look like when L is a purely inseparable extension of K?
(4) Let A be noetherian and integrally closed in its field of fractions K, let L be a finite separable extension of K, and let B be the integral closure of A in L. Show that B is finitely generated as an A-module.
(5) Compute the integral closure of $\mathbb{F}_p[t] \subseteq \mathbb{F}_p(t)$ in $\mathbb{F}_p(t^{1/p})$.

2. Number theory

2.1. **Algebraic number theory: global theory.**

(1) Let $K = \mathbb{Q}(\sqrt{3})$. Describe its ring of integers, discriminant, and ramification and splitting of primes (in terms of congruences). Exhibit the smallest m such that K is a subfield of the cyclotomic field $L = \mathbb{Q}(\zeta_m)$, and then show that $L \supseteq K$ is unramified at all (finite) primes of K—but that L is ramified over K at each of the infinite places. Does K have any everywhere unramified abelian extension?
(2) Give several equivalent definitions of a Dedekind domain. State which of the following are Dedekind domains:

$$k[[t]] \ (k \text{ a field}), \quad \{a/b \in \mathbb{Q} : 3 \nmid b\}, \quad \mathbb{C}[x,y]/(y^2 - x^3).$$

(3) A Dedekind domain is noetherian, integrally closed, and Krull dimension 1 (or ≤ 1 if you prefer). Give examples of domains that satisfy exactly two of the three (but not the third) in each of the three cases.
(4) Compute as efficiently as you can the ring of integers of $\mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$ is a fundamental discriminant.
(5) Let $K = \mathbb{Q}(\alpha)$ where $\alpha^3 - \alpha + 1 = 0$. Show that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ and $\mathrm{disc}(K) = -23$. Then compute the Galois closure L of K, and show that only 23 and ∞ ramify in $L \supseteq \mathbb{Q}$. Finally, show that L is unramified over $F = \mathbb{Q}(\sqrt{-23})$; what does that tell you about the class group of F and primes of the form $x^2 + xy + 6y^2$?

(6) Let $K = \mathbb{Q}(\alpha)$ where $\alpha^3 - 2\alpha - 2 = 0$. Show that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ and $\operatorname{disc}(K) = -76$. What primes ramify in K, and how do they factor in $\mathbb{Z}_K$? Show that $\mathbb{Z}_K$ has class number 1.

(7) Let K be a Galois number field and p a prime unramified in K. How do you define the conjugacy class Frob_p? If $K = \mathbb{Q}(\alpha)$ where $f(\alpha) = 0$ and $p \mid \operatorname{disc}(f)$, how does Frob_p relate to the factorization of f modulo p? What does it mean when Frob_p is trivial? How do you define the decomposition group, and what is its relationship to Frob_p?

(8) Define decomposition group and inertia group and relate them by an exact sequence. What are the corresponding orders of these groups in terms of the fundamental invariants? What is the characterizing property of the corresponding fixed field ("they are the largest/smallest subfields where ...")?

(9) Let p, q be distinct primes. (You may take $p = 2$ to begin, if you need to.) Let $K = \mathbb{Q}(\zeta_p, \sqrt[p]{q})$. What are the primes (of $\mathbb{Q}$) that ramify in K? What are the inertia groups for these primes?

(10) Define the class group of a Dedekind domain. Compute the class group of $K = \mathbb{Q}(\sqrt{-6})$. Once you know it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, compute the Hilbert class field by showing that $L = K(\sqrt{-3})$ is unramified over K (including at ∞). What are the possible ramification and splitting behaviors of primes in the extension $L \supseteq \mathbb{Q}$?

(11) Let p be an odd prime. How do primes factor in $\mathbb{Q}(\zeta_p)$ (according to their congruence class modulo p)? Prove quadratic reciprocity in the form

$$\left(\frac{p^*}{q}\right) = \left(\frac{q}{p}\right), \quad \text{where } p^* = \left(\frac{-1}{p}\right) p$$

for primes $q \neq p$ using this factorization and the fact that $\mathbb{Q}(\sqrt{p^*}) \subseteq \mathbb{Q}(\zeta_p)$.

(12) Prove the supplements to quadratic reciprocity using (subfields of) cyclotomic fields.

(13) Show that if K, L are number fields with coprime discriminants $\gcd(d_K, d_L) = 1$, then $K \cap L = \mathbb{Q}$. (How do you know that if $F \supseteq \mathbb{Q}$ has $d_F = 1$ then $F = \mathbb{Q}$?)

(14) Show that if $K \supseteq F \supseteq \mathbb{Q}$ is a tower of number fields, then

$$d_K = d_F^{[K:F]} \operatorname{Nm}_{F|\mathbb{Q}}(\mathfrak{d}_{K|F}).$$

Show that if K has d_K squarefree then K has no nontrivial subfields.

(15) What is the Hilbert class field of $\mathbb{Q}(\sqrt{10})$?

(16) Show that $\mathbb{Z}[i]$ is a Euclidean domain.

(17) Show that a PID is a Dedekind domain, and that a Dedekind domain is a PID if and only if it is a UFD.

2.2. **Algebraic number theory: local theory.**

(1) Show that $\mathbb{Z}_p$ is a DVR.

(2) Define $\mathbb{Z}_p$ as a projective limit. What is the topology on $\mathbb{Z}_p$ (Hausdorff, compact, and totally disconnected)? Prove that $\mathbb{Z}_p$ is compact and totally disconnected. Give a basis of neighborhoods of 0, and show that open balls are closed.

(3) Find all pure cubic extensions of $\mathbb{Q}_7$. What does this tell you about the abelian extensions?

(4) Exhibit the archimedean places and the places above 2 in the following fields:

$$\mathbb{Q}, \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$$

(Draw a diagram.)

(5) Prove the fundamental identity $n = ef$ for a separable extension $L \supseteq K$ of local fields, where e is the ramification degree and f is the inertial degree. Use this to prove the fundamental identity $n = \sum_{i=1}^r e_i f_i$ for an extension of global fields.
(6) Prove that the compositum of two unramified extensions of a local field is unramified.
(7) Show that there is a unique unramified extension of a local field in each degree, and show the relative trace and norm are surjective in these extensions.
(8) Give an explicit polynomial $f(x) \in \mathbb{Q}_3[x]$ of degree 4 such that $K = \mathbb{Q}_3[x]/(f(x))$ is a totally ramified quartic extension of $\mathbb{Q}_3$.
(9) Let p be prime and $n \in \mathbb{Z}_{\geq 1}$ coprime to p. Show that the map $1 + p\mathbb{Z}_p \to 1 + p\mathbb{Z}_p$ given by $x \mapsto x^n$ is an isomorphism of topological groups.
(10) How do you describe the topology on $\mathbb{Z}_p$? (Hausdorff, compact, totally disconnected, like all profinite groups; homeomorphic to a Cantor set.) Show that there is a continuous surjection $\mathbb{Z}_p \to [0, 1]$; is there a continuous surjection in the other direction?

2.3. **Analytic number theory.**

(1) Factor the Dedekind zeta function of a quadratic field as the product of two degree 1 L-functions.
(2) What is a Dirichlet character mod N? What is N called? When is it primitive?
(3) What is the prime number theorem, and how does it relate to the zeros of the Riemann zeta function $\zeta(s)$? (What are the trivial zeros of $\zeta(s)$?)
(4) What is the functional equation for ζ and how do you prove it using Poisson summation?
(5) Recognize the series $\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (-1)^{(n-1)/2}/n^s$ as a Dirichlet L-series. What are its analytic properties, and what is the value at $s = 1$? How does this relate to the Dedekind zeta function of $\mathbb{Q}(i)$?
(6) What is the Chebatorev density theorem? Use it to deduce Dirichlet's theorem on primes in arithmetic progression.

3. Algebraic geometry

3.1. **Varieties.**

(1) Consider the image Y of the parametrization

$$\begin{aligned} \mathbb{A}^1 &\to \mathbb{A}^3 \\ t &\mapsto (t, t^2, t^3). \end{aligned}$$

Show that Y is an affine variety and find its defining ideal. (How do you compute the image in general?) What is the projective closure $\overline{Y}$ (the *twisted cubic curve*)? It may help to consider the homogeneous parametrization

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ [s : t] &\mapsto [s^3 : s^2t : st^2 : t^3] \end{aligned}$$

and compute its defining ideal $I(\overline{Y})$. With respect to a few nice term orders, compute the leading terms and initial ideal of $I(\overline{Y})$. What is the Hilbert function and Hilbert polynomial of $I(\overline{Y})$? What do its coefficients tell you?

(2) Consider the quadric surface

$$Q : xw - yz = 0$$

in $\mathbb{P}^3$. Show that it contains the twisted cubic curve. Does Q contain any lines? How are they parametrized? Show that we have an isomorphism of varieties $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Is it true that $\mathbb{P}^1 \times \mathbb{P}^1 \simeq \mathbb{P}^2$?

(3) Show that an affine variety is a projective variety if and only if it is a point.

(4) Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over an algebraically closed field (or more generally, just a commutative ring). For an ideal $\mathfrak{a} \subseteq R$, define $V(\mathfrak{a})$ and show that $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$. Similarly, given a subset $Z \subseteq k^n$ (or more generally, $Z \subseteq \operatorname{Spec} R$), define $I(Z)$ and show that $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ and $V(I(Z)) = \operatorname{cl}(Z)$ is the closure of Z under the Zariski topology.

(5) Show that the curve $X\colon y^2 = x^3 + x^2$ over a field k has a singular point at $P = (0, 0)$. What are the nicest description you can give of the local ring and the completion of the local ring at P? Compute the normalization of X and describe the birational map between X and its normalization.

3.2. **Sheaves and schemes.**

(1) Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups on a topological space X. What does this mean? Let $P \in X$. Show that $\mathscr{K} = \ker \phi$ is a subsheaf of $\mathscr{F}$, and that $(\ker \phi)_P = \ker(\phi_P)$.

(2) Let X be a scheme and $Z \subseteq X$ and irreducible closed subset. Show that Z corresponds to a generic point of X. (Without loss of generality, you can reduce to the affine case.) Suppose that X is an integral scheme. Show that its function field is the local ring at its generic point.

(3) Is an integral scheme irreducible?

(4) What is a morphism of finite type? What does this mean for X to be a scheme of finite type over $\operatorname{Spec} k$ where k is a field—what do the affine opens look like? What are their closed points if k is algebraically closed? Give a couple of examples of schemes over $\operatorname{Spec} k$ which are not of finite type.

(5) Let X be a scheme. What is a quasi-coherent $\mathscr{O}_X$-module? A coherent $\mathscr{O}_X$-module?

(6) Let X be a scheme and let $\mathscr{F}, \mathscr{G}$ be an $\mathscr{O}_X$-modules. Define the sheaf $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$. (Why is the presheaf a sheaf?) If $\mathscr{F}, \mathscr{G}$ are coherent, is $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$?

(7) What is an example of a sheaf on a scheme that is not quasi-coherent? *[Hint: extend by zero.]*

(8) Let X be a scheme and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Show that there is a natural map $\mathscr{O}_X \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})$; we call the kernel the *annihilator*. What is the support of the annihilator? If $\mathscr{F}$ is coherent, show that the annihilator is coherent; what does this tell you? How do you interpret this directly in terms of modules in the case where X is affine and finite type over $\mathbb{C}$?

(9) Let X be a topological space, let A be a nonempty set, and define the *constant presheaf* on X by $U \mapsto A$. Give an example where this presheaf is not a sheaf. What is the sheafification in general—can you define it intrinsically?

3.3. **Elliptic curves.**

(1) What is a Weierstrass equation for an elliptic curve? Why does every elliptic curve have a Weierstrass equation? If we equip a genus 1 curve C with a divisor of degree 2, what kind of equation do you get from this kind of argument?
(2) Let E be the elliptic curve over $\mathbb{Q}$ defined by $y^2 + y = x^3 - x^2$. What are the primes of bad reduction? For each prime of bad reduction, what is its reduction type?
(3) Let E be the elliptic curve over $\mathbb{Q}$ defined by $y^2 + y = x^3 - x^2$; this curve has good reduction at $p = 2$. On this reduction, what is the characteristic polynomial of the Frobenius endomorphism?