Proper Actions and Groupoid Equivalence

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1 Proper Actions and Equivalence

Scott LaLonde asked the following question. "If two second countable groupoids are equivalent, must the space implementing the equivalence be second countable." This question is not addressed in the basic papers on the subject, although a "yes" answer seems to have implicitly been assumed in a number of instances. Fortunately, the answer is yes. This is a consequence of the following theorem.

Theorem 1.1. *Suppose that G is a second countable, locally compact Hausdorff groupoid acting freely and properly on a locally compact space P such that the orbit space $G\backslash P$ is second countable. Then P is second countable.*

Remark 1.2 (Necessary Hypotheses). I am certain that we should be able to replace "Hausdorff" with "locally Hausdorff", but have not pursued that as yet. My gut tells me that we also should be able to drop "free" and even "proper", but the proof below makes significant use of both. In fact, I had to work a bit to find even the solution given here. Nevertheless, I feel that this is not "the right" proof. I would most definitely appreciate seeing any "better" proofs — even in the special cases of *group actions* or even for *group extensions* (see Corollaries 1.6 and 1.7).

Of course, we get an answer to Scott's question as an immediate corollary.

Corollary 1.3. *Suppose that P implements an equivalence between two second countable groupoids. Then P is second countable.*

We need some preliminaries. We'll write $\pi : P \to G\backslash P$ for the orbit map.

Lemma 1.4. *If P is a free and proper G-space with G and $G\backslash P$ second countable, then P is σ-compact. If fact, if $U \subset G^{(0)}$ is open, then $r_P^{-1}(U)$ is σ-compact.*

Proof. It suffices to prove the second statement. Since $G\backslash P$ is second countable and locally compact, it is σ-compact. Say $G\backslash P = \bigcup C_n$ with each C_n compact. Since the orbit map is continuous and open, there is a compact set $K_n \subset P$ such that $\pi(K_n) = C_n$. Since G is σ-compact, so is $r_G^{-1}(U) \subset G$. Hence,

$$r_P^{-1}(U) = \bigcup r_G^{-1}(U) \cdot K_n$$

is σ-compact as claimed. □

The next lemma is key. Unfortunately, is strongly uses the properness and the freeness of of the action. We'll adopt Palais's notation so that if K is a subset of P, then

$$(K, K) := \{ x \in G : x \cdot K \cap K \neq \emptyset \}.$$

Lemma 1.5. *Given $p \in P$ and a neighborhood V of $r(p)$ in G, there is pre-comapct open neighborhood W of p such that*

$$(\overline{W}, \overline{W}) \subset V.$$

Proof. If no such neighborhood exists, then for every pre-compact open neighborhood W of p there is $x_W \notin V$ and $p_W \in \overline{W}$ such that $x_W \cdot p_W \in \overline{W}$. Clearly both the nets $\{p_W\}$ and $\{x_W \cdot p_W\}$ converge to p. The freeness and properness of the action forces $x_W \to r(p)$.[1] Of course, that leads to a contradiction as any limit of $\{x_W\}$ must lie in the (closed) complement of V. □

If X is a second countable, locally compact Hausdorff space, then we'll call a countable neighborhood basis $\{B_n\}_{n=1}^{\infty}$ at x a *regular neighborhood basis* if each B_n is open and precompact with $\overline{B_{n+1}} \subset B_n$. Of course, if β is a basis for the topology on X, then β contains a regular neighborhood basis of each point in x.

[1] If K is a compact neighborhood of p, then eventually $\{x_W\} \subset (K, K)$. If the action is proper, (K, K) is compact. Thus $\{x_W\}$ has a subnet converging to x. But $x \cdot p = p$ forces $x = r(p)$. Now every subnet of has a subnet converging to $r(p)$, hence the assertion.

Proof of Theorem 1.1. Let $\mathscr{E}$ be a countable collection of pre-compact open sets in G consisting of those elements of a fixed basis for the topology on G that have nontrivial intersection with $G^{(0)}$. Then $\mathscr{E}$ contains a regular neighborhood basis in G of every $u \in G^{(0)}$. If $V \in \mathscr{E}$ and $p \in r_P^{-1}(V \cap G^{(0)})$, then Lemma 1.5 implies there is a pre-compact open neighborhood W of p such that $(\overline{W}, \overline{W}) \subset V$. But Lemma 1.4 implies that $r_P^{-1}(V \cap G^{(0)})$ is σ-compact (hence Lindelöf).[2] Thus there is a countable collection $\mathscr{D}_V$ of pre-compact open sets W that cover $r_P^{-1}(V \cap G^{(0)})$ with the property that $(\overline{W}, \overline{W}) \subset V$. Since $\mathscr{E}$ is countable, there is a countable collection $\mathscr{D}$ of pre-compact open sets W in P such that given $V \in \mathscr{E}$ and $p \in r_P^{-1}(V \cap G^{(0)})$ there is a $W \in \mathscr{D}$ such that $p \in W$ and $(\overline{W}, \overline{W}) \subset V$. We can even assume that $\mathscr{D}$ is closed under finite intersections.

Let β be a countable basis for $G \backslash P$. We aim to show that the countable collection of open sets

$$U = \pi^{-1}(B) \cap W$$

with $B \in \beta$ and $W \in \mathscr{D}$ is a basis for the topology on P.

To this end, let O be an open neighborhood of p in P. Let $\{V_n\} \subset \mathscr{E}$ be a regular neighborhood basis of $r(p)$ in G, and let $\{B_n\}$ be a regular neighborhood basis of $\pi(p)$ in $G \backslash P$. Let $W_n \in \mathscr{D}$ such that $p \in W_n$ and $(W_n, W_n) \subset V_n$. Using finite intersections, we can assume that $W_{n+1} \subset W_n$.

Now let

$$U_n := \pi^{-1}(B_n) \cap W_n.$$

Note that $U_{n+1} \subset U_n$. It will suffice to see that there is a $n \in \mathbf{N}$ such that $p \in U_n \subset O$. If not, then for each $n \in \mathbf{N}$ there is

$$q_n \in U_n \setminus O.$$

Then $\{q_n\}$ is in the compact set $\overline{W_1}$. Then there must be a convergent sub*net*,[3] say $\{q_{n_i}\}_{i \in I}$ converging to $q \in P \setminus O$. Note that given $n \in \mathbf{N}$, there is an $i_0 \in I$ such that $i \geq i_0$ implies $n_i \geq n$. Hence $\pi(q) \in \overline{B_n}$ for all n. That forces $\pi(q) = \pi(p)$. Therefore $q = x \cdot p$ for some $x \neq r(p)$ (since $q \neq p$). But then there is a m such that $x \notin V_m$. But as above, then q_{n_i} are eventually in W_m. Hence $q \in \overline{W_m}$. But then $x \in (\overline{W_m}, \overline{W_m}) \subset V_m$ which is a contradiction. $\square$

[2] Recall that a space is called Lindelöf if every open cover has a countable subcover. Clearly, σ-countable spaces are Lindelöf spaces.

[3] I haven't bothered to show that P is first countable yet — hence the annoyance of using nets.

1.1 Some Corollaries and Remaining Questions

Here are some immediate corollaries of Theorem 1.1.

Corollary 1.6. *Suppose that G is a second countable, locally compact group acting freely and properly on a locally compact space P with $G\backslash P$ second countable. Then P is second countable.*

For the definition of a short exact sequence of topological groups, see [Wil07, Definition 1.51].

Corollary 1.7. *Let*

$$e \longrightarrow H \longrightarrow E \longrightarrow G \longrightarrow e \tag{1.1}$$

be short exact sequence of locally compact groups. Then E is second countable if and only if H and G are.

However, the assumption of local compactness is not necessary in Corollary 1.7. We have the following.

Lemma 1.8 ([HR63, II.8.19])**.** *Let (1.1) be an extension of topological groups.*[4] *Then E is second countable if and only if H and G are.*

Proof. We can assume that H is a closed subgroup of E and that $G = E/H$. If E is second countable, then H is. Since the quotient map $\pi : E \to E/H$ is continuous and open, E/H is also second countable.

So assume H and E/H are second countable. It is fairly easy to see that E must be separable: let $\{g_i\}$ be a set of representatives for a countable dense subset of E/H. Then if $\{h_i\}$ is a countable dense set in H, consider $\{g_i h_j\}$. Therefore to see that E is second countable, it suffices to see that E is metrizable. But Theorem 2.2 implies that it suffices to find a countable neighborhood basis for e in E.[5]

However, just as in the proof of Theorem 1.1, given any open set V in H, there is a neighborhood W of e in E such that

$$(W, W) := \{\, g \in E : gW \cap W \neq \emptyset \,\} \subset V.$$

[4] I'm assuming the definition of topological group includes the fact that points are closed and hence that the groups themselves are Hausdorff. See [Wil07, Definition1.1] and [Wil07, Lemma 1.13].

[5] The proof of Theorem 2.2 is much better than the standard reference [HR63, Theorem II.8.3] as it constructs the metric in a very concrete way. The proof of Theorem 2.2 given here is due to Moore. I took it from class notes.

(If not, then for every neighborhood W, there is a $g_W \in E$ and $h_W \notin V$ such that g_W and $g_W h_W$ both belong to W. Clearly $g_W \to e$ and $g_W h_W \to e$. Hence $h_W \to e$, leading to a contradiction.) Thus if $\{V_n\}$ is a countable neighborhood basis for e in H, then we can form neighborhoods W_n of e in E such that

$$(W_n, W_n) \subset V_n.$$

There is no harm in assuming that $V_{n+1} \subset V_n$ and $W_{n+1} \subset W_n$. Now let $\{B_n\}$ be a similar neighborhood basis of eH in E/H.

$$U_n := \pi^{-1}(B_n) \cap W_n.$$

I claim $\{U_n\}$ is the desired basis of e in E. If not, then there is a neighborhood U of e in E and a $g_n \in U_n \setminus U$ for all n. Clearly, $\pi(g_n) \to eH$ in E/H. Since π is open, we can pass to a subnet, $\{g_{n_i}\}$ and find $h_{n_i} \in H$ such that $g_{n_i} h_{n_i} \to e$ in E.

Now fix n. We eventually have $g_{n_i} h_{n_i}$ and g_{n_i} in W_n. Hence we eventually have $h_{n_i} \in V_n$. That is, $h_{n_i} \to e$. But then $g_{n_i} \to e \in U$. This is a contradiction and completes the proof. $\square$

2 Separable Metric Groups and Completions

Remark 2.1. Now let (1.1) be a short exact sequence of topological groups with H and G Polish. Must E be Polish? In [Bro71], Brown claims this is obvious. But other than seeing that E is second countable and metrizable (by Lemma 1.8), I was stuck. The purpose of this section is to provide a proof. The material is based on Cal Moore's course on topological groups I took back in the late 1970s. The lectures no doubt were based on [Moo76] where the result appears as Proposition 3. Moore does attribute the result to Brown.

This is Moore's version of the standard result in Hewitt & Ross [HR63, Theorem II.8.3]. It is considerably more concrete than their result. In so far as I know, this construction does not appear in print.

Theorem 2.2. *A topological group G is metrizable if and only if there is a countable base for the topology at e. (Since G is a group, it has a countable base at e if and only if it is first countable.)*

Proof. If G is metrizable, then it is first countable and the result is clear.

Suppose that $\{U_n\}$ is a countable base at e. We can assume $U_n^{-1} = U_n$ and that $U_{n+1}^3 \subset U_n$. We let $U_0 = G$.

Then, for each $g \in G$, define

$$\|g\| = \inf\{\, 2^{-n} : g \in U_n \,\}.$$

Note that $\|g\| = 0$ if and only if $g = e$ (since $\bigcap U_n = e$), and that $\|g\| = 2^{-n}$ if $g \in U_n \setminus U_{n+1}$. Then let

$$|g| = \inf\Big\{ \sum_{i=1}^{k} \|g_i\| : g = g_1 g_2 \cdots g_k \Big\}.$$

I claim that

$$\begin{align} |gh| &\le |g| + |h|, \tag{2.1} \\ |g| &\le \|g\| \quad \text{and} \tag{2.2} \\ \|g\| &\le 2|g|. \tag{2.3} \end{align}$$

Equations (2.1) and (2.2) are clear. To establish (2.3), it will suffice to show that $g = g_1 \cdots g_k$ implies that $\|g\| \le 2 \sum_{i=1}^{k} \|g_i\|$. Of course, we can assume that $g \ne e$.

This is easy if $k = 1$, so we proceed by induction. Let $k > 1$ and assume that the result holds for products of $k-1$ or fewer elements. Let

$$a := \sum_{i=1}^{k} \|g_i\|.$$

Let l be the largest index such that $\sum_{i=1}^{l} \|g_i\| \le a/2$. (Note: we could have $l = 0$ if $\|g_1\| > a/2$.) Anyway, we have

$$a = \sum_{i=1}^{l} \|g_i\| + \|g_{l+1}\| + \sum_{i=l+2}^{k} \|g_i\|.$$

Using the induction hypotheses,

$$\Big\|\prod_{i=1}^{l} g_i\Big\| \le 2\Big(\sum_{i=1}^{l} \|g_i\|\Big) \le a \quad \text{and} \quad \Big\| \prod_{i=l+2}^{k} g_i\Big\| \le 2\Big(\sum_{i=l+2}^{k} \|g_i\|\Big) \le a.$$

Note that $\|h\| \le a$ implies that $h \in U_n$ for all n with $a \le 2^{-n}$. Let n be the largest integer such that $a \le 2^{-n}$. Then the above forces each of $\prod_{i=1}^{l} g_i$, g_{l+1} and $\prod_{i=l+2}^{k} g_i$

to belong to U_n. But $U_n^3 \subset U_{n-1}$. Hence $g = \left(\prod_{i=1}^{l} g_i\right)(g_{l+1})\left(\prod_{i=l+2}^{k} g_i\right) \in U_{n-1}$. Hence $\|g\| \leq 2a$ as was to be proved.

Now we define $\rho(g,h) = |g^{-1}h|$. It is easy to see that ρ is a metric on G. Firstly, in view of (2.3), $\rho(g,h) = 0$ implies that $\|g^{-1}h\| = 0$ and that clearly implies $g = h$. The triangle inequality follows from (2.1).

It only remains to see that the ρ-topology is the original topological group topology. But consider the ρ-ball $B := B^{\rho}_{2^{-n}}(e)$. If $g \in U_n$, then $\|g\| \leq 2^{-n}$. But then $|g| \leq 2^{-n}$ and $g \in B$. But if $g \in B$, then $\|g\| \leq 2|g| < 2^{n-1}$ and $g \in U_{n+1}$. That is, $U_{n+1} \subset B \subset U_n$ and it follows that the topologies are the same. □

Example 2.3. If G is discrete, then we can let $U_n = \{e\}$ for all $n \geq 1$. Then we get the "usual" discrete metric: $\rho(x,y) = 1 - \delta_{x,y}$.

Corollary 2.4. *If a topological group G is metrizable, then it admits a left-invariant (resp., right invariant) metric. With respect to* any such metric*, a sequence $\{g_i\}$ is Cauchy if and only if given any neighborhood V of e in G there is a N such that $i,j \geq N$ implies $g_i^{-1}g_j \in V$ (resp., $g_i g_j^{-1} \in V$).*

Proof. The metric we construct in Theorem 2.2 is left-invariant. But if ρ is any left invariant metric, then $\rho(g_i, g_j) = \rho(e, g_i^{-1}g_j)$ and the rest follows from this. Producing right invariant versions requires only the obvious alterations in the proof of Theorem 2.2. □

Remark 2.5. Note that the criteria mentioned in the above corollary applies to any left (or right) invariant metric — not just the one we constructed. Furthermore, if G is not abelain or compact, one can't expect to always find a bi-invariant metric.[6] See [HR63, II.8.18] for criteria for the existence of bi-invariant metrics.

For locally compact groups, the criteria for metrizability has another nice form. The regularity is crucial — the result is apparently not true without something like local compactness.

Corollary 2.6. *A locally compact group admits a metric if and only if e is a G_δ subset of G. (Because G is a group, this is equivalent to saying that points in G are G_δ sets.)*

[6] If G has a bi-invariant metric ρ, then given a neighborhood V of e in G, there is a neighborhood M of e such that $aMa^{-1} \subset V$ for *all* $a \in G$. But if $G = \mathrm{GL}_2(\mathbf{R})$, then given any $\epsilon > 0$ and $R > 0$, there are $B, A \in G$ such that $\|B - I\| < \epsilon$ and $\|ABA^{-1} - I\| > R$.

Proof. Suppose that there are open sets O_n such that $\{e\} = \bigcap O_n$. (The other direction being clear.) By regularity, we can find pre-compact open sets U_n such that $e \in U_n \subset \overline{U_n} \subset O_n$. There is then no harm in assuming that $U_{n+1} \subset U_n$. Then, in view of Theorem 2.2, we just have to see that $\{U_n\}$ is a basis at e. Let U be any neighborhood of e. If no U_n is contained in U, then pick $g_n \in U_n \setminus U$. Since all the g_n are in the compact set $\overline{U_1}$, we have a convergent subnet $\{g_{n_i}\}$ converging to $g \notin U$. But the g_{n_i} are eventually in the closure of each U_n. Hence $g = \bigcap O_n = \{e\}$ which is a contradiction. □

Lemma 2.7. *Let (G, ρ) be a metric group equipped with a left-invariant metric. If $\{x_i\}$ and $\{y_i\}$ are Cauchy sequences, then so is $\{x_i y_i\}$.*

Proof. Let V be a symmetric neighborhood of e. As in Corollary 2.4, we need to find N such that $i, j \geq N$ implies $y_i^{-1} x_i^{-1} x_j y_j \in V$. Let U be a symmetric neighborhood of e such that $U^3 \subset V$. Let M be such that $i, j \geq M$ implies $y_i^{-1} y_j \in U$. We have

$$y_i^{-1} x_i^{-1} x_j y_j = (y_i^{-1} y_M)(y_M^{-1} x_i^{-1} x_j y_M)(y_M^{-1} y_j). \tag{2.4}$$

Let M' be such that $i, j \geq M'$ implies $x_i^{-1} x_j \in y_M U y_M^{-1}$. Then if $i, j \geq N = \max\{ M, M' \}$ we have the right-hand side of (2.4) in $U^3 \subset V$ as required. □

Let ρ be a left-invariant metric. Unfortunately, examples show that if $\{g_n\}$ is ρ-Cauchy, then $\{g_n^{-1}\}$ need not be ρ-Cauchy (see [Num80, Example 4]). Of course that same is true for right invariant metrics λ.[7] This means we'll want to work with symmetric metrics of the form $\sigma = \rho + \lambda$ where ρ is left-invariant and λ is right-invariant. Then we get the following as an easy consequence of Corollary 2.4.

Lemma 2.8. *Let $\sigma = \rho + \lambda$ be the a symmetric metric on G. Then a sequence $\{g_n\}$ is σ-Cauchy if and only if given a neighborhood V of e in G, there is a N such that $i, j \geq N$ implies both $g_i^{-1} g_j \in V$ and $g_i g_j^{-1} \in V$. In particular, if $\{g_i\}$ is σ-Cauchy, then so is $\{g_i^{-1}\}$.*

At this point, we fix a symmetric metric $\sigma = \rho + \lambda$. Now we let $(\mathfrak{G}, \mathfrak{s})$ be the metric space completion of (G, σ) (see [Wil70, Theorem 24.4]). Recall that the elements of $\mathfrak{G}$ are equivalence classes $\mathfrak{a} = [a_n]$ of σ-Cauchy sequences (a_n) in G where $(a_n) \sim (b_n)$ if $\lim_n \sigma(a_n, b_n) = 0$. Then $\mathfrak{s}(\mathfrak{a}, \mathfrak{b}) = \lim_n \sigma(a_n, b_n)$ is a well-defined complete metric on $\mathfrak{G}$. (In particular, the limit exists.) There is an obvious topological embedding $k : G \to \mathfrak{G}$ sending $g \in G$ to the constant sequence. (In fact, the embedding is isometric!) If $\mathfrak{g} = (g_n)$, then $k(g_n) \to \mathfrak{g}$ in $\mathfrak{G}$.

[7] Of course, I got the notation backwards, but I'm not switching now.

If V is a neighborhood of e in G, then let

$$D_V^G := \{\, (x,y) \in G \times G : x^{-1}y \in V \text{ and } xy^{-1} \in V \,\}.$$

A function $f : G \to H$ is said to be *uniformly continuous* with respect to the symmetric metrics if given a neighborhood W of e in H, there is a neighborhood V of e in G such that $(x,y) \in D_V^G$ implies $\big(f(x), f(y)\big) \in D_W^H$.

Theorem 2.9. *Let $\mathfrak{G}$ be the completion of G as above. Then $\mathfrak{G}$ is a group with respect to the operations $\mathfrak{a} \cdot \mathfrak{b} = [a_n b_n]$ and $(\mathfrak{a})^{-1} = [a_n^{-1}]$. Furthermore, G is dense in $\mathfrak{G}$ and if $j : G \to H$ is any homomorphism into a* complete *metric group which is* uniformly continuous with respect to the symmetric metrics *on G and H, then there is a unique continuous homomorphism $\bar{j} : \mathfrak{G} \to H$ extending j. In fact, if j is any uniformly continuous function from G to H, then there is a unique continuous extension $\bar{j}$.*

Lemma 2.10. *The group operations are well defined.*

Proof. The operations are well-defined once we observe that the product given by Lemma 2.7 and the inverse given by Lemma 2.8 respect equivalence classes. (I didn't check this carefully.) □

Lemma 2.11. *Multiplication is continuous on $\mathfrak{G}$.*

Proof. Since a σ-Cauchy sequence is also ρ-Cauchy and λ-Cauchy, we can equally well define $\overline{\rho}$ and $\overline{\lambda}$. In fact, $\overline{\sigma} = \overline{\rho} + \overline{\lambda}$. By symmetry, it will suffice to see that multiplication is $\overline{\rho}$ continuous. Suppose that $\mathfrak{a}_n \to \mathfrak{a}$ and $\mathfrak{b}_n \to \mathfrak{b}$. We need to see that

$$\overline{\rho}(\mathfrak{a}_n\mathfrak{b}_n, \mathfrak{a}\mathfrak{b}) = \lim_{k\to\infty} \rho(a_{n,k}b_{n,k}, a_k b_k)$$

tends to zero with n. Thus given any neighborhood V of e in G it will suffice to see that there is a N_V such that $n \geq N_V$ implies that

$$b_{n,k}^{-1}a_{n,k}^{-1}a_k b_k \tag{2.5}$$

is k-eventually in V.

Let U be a neighborhood of e in G such that $U^4 \subset V$. Since $\{b_k\}$ is ρ-Cauchy, there is a M such that $k, m \geq M$ implies that $b_m^{-1}b_k \in U$. Since $\mathfrak{b}_n \to \mathfrak{b}$, there is a N_1 such that $n \geq N_1$ implies that $b_{n,k}^{-1}b_k$ is k-eventually in U. Similarly, there is N_2 such that $n \geq N_2$ implies that $a_{n,k}^{-1}a_k$ is eventually in $b_M U b_M^{-1}$.

Now if $n \geq N := \max\{ N_1, N_2 \}$, then there is a $K \geq M$ such that $k \geq K$ implies that

$$b_k^{-1} b_M, b_M^{-1} b_k, b_{n,k}^{-1} b_k \in U \quad \text{and} \quad a_{n,k}^{-1} a_k \in b_M U b_M^{-1}.$$

But then (2.5) is equal to

$$(b_{n,k}^{-1} b_k)(b_k^{-1} b_M)(b_M a_{n,k}^{-1} a_k b_M)(b_M^{-1} b_k) \in U^4 \subset V.$$

This completes the proof. □

Lemma 2.12. *Inversion is continuous on $\mathfrak{G}$.*

Proof. It is not hard to see that $\lim_n \overline{\rho}(\mathfrak{b}_n, \mathfrak{b}) = 0$ if and only if $\lim_n \overline{\lambda}(\mathfrak{b}_n^{-1}, \mathfrak{b}^{-1})$. This suffices. □

Proof of Theorem 2.9. We now have continuous well-defined operations. To see that $\mathfrak{G}$ is indeed a topological group, we proceed as follows. Note that the maps $(x, y, z) \mapsto (xy)z$ and $(x, y, z) \mapsto x(yz)$ agree on the dense set $G \times G \times G$. Hence they agree on $\mathfrak{G} \times \mathfrak{G} \times \mathfrak{G}$. Similarly $e[a] = [a]e = [a]$, etc.

We still need to see the universal property holds. Let $j : G \to H$ be a uniformly continuous map into a complete metric group H. By Proposition 2.13, the symmetric metric is complete on H. But if $[a_n] \in \mathfrak{G}$, then the uniform continuity of j implies that $\{j(a_n)\}$ is Cauchy in H with respect to any symmetric metric. Hence the limit, $\lim_n j(a_n)$ exists. We simply define $\bar{j}([a_n]) = \lim_n j(a_n)$. This map obviously extends j and is a homomorphism if j is. We just have to prove continuity.

Suppose that $\mathfrak{g}^n \to \mathfrak{g}$ in $\mathfrak{G}$. Let $h^n = \bar{j}(\mathfrak{g}^n)$ and $h = \bar{j}(\mathfrak{g})$. It will suffice to show that given a neighborhood U of e in H, there is a N such that $n \geq N$ implies $h^n h^{-1} \in U$. Let V be a neighborhood of e in H such that $V^3 \subset U$. Since j is uniformly continuous there is a neighborhood W of e in G such that $xy^{-1} \in W$ implies that $j(x)j(y)^{-1} \in V$. Since, by assumption,

$$\lim_{n\to\infty} \lim_{k\to\infty} \sigma(g_k^n, g_k) = 0,$$

there is a N such that $n \geq N$ implies there is K_n such that $k \geq K_n$ implies $g_k^n g_k^{-1} \in W$. Now if $n \geq N$, we can find $K \geq K_n$ such that $h^n j(g_k^n) \in V$ and $j(g_k)h^{-1} \in V$. But then

$$h^n h^{-1} = \big(h^n j(g_k^n)^{-1}\big)\big(j(g_k^n) j(g_k)^{-1}\big)\big(j(g_k)h^{-1}\big) \in V^3 \subset U.$$

This completes the proof once we prove the following assertion. □

The key observation for the proof of Brown's assertion that the extension of Polish groups is Polish is the following. In fact, this in essence is [Moo76, Proposition 1].

Proposition 2.13. *If G is a polish group, then G is complete with respect to any symmetric metric σ.*

Remark 2.14. In other words, G admits a complete metric if and only if σ is already complete. Moore introduces this[8] at the very beginning of [Moo76] — it is the content of his Proposition 1. However, as proof, he merely cites [Kel55, p. 211]. However, if you bother to look up [Kel55, p. 211] there is just the statement without proof (on page 212). Fortunately, Kelly cites [Kle52]. Unfortunately, Klee doesn't actually prove the statement Kelly gives. But Klee does prove something similar for bi-invariant metrics; a responsible reference would say that Klee's proof for bi-invariant metrics carries over to symmetric metrics quite easily.

The proof relies on some basic topological category stuff. Note that any subset of a first category subset of a space is itself of first category of the total space.

Lemma 2.15. *Suppose that G is a second category topological group and that H is a subgroup. Then the complement of H in G is either empty or of second category in G. In particular, if H is a dense G_δ-subset of G, then $H = G$.*

Proof. Let H^C be the complement of H in G. Suppose there is a $g \in H^C$. Then $gH \subset H^C$. Thus if H^C is of first category in G, then so is gH. Hence H is of first category as is $G = H \cup H^C$. This proves the first assertion.

If $H = \bigcap_{i=1}^{\infty} O_i$ with each O_i dense and open, then $G \setminus O_i$ is closed and nowhere dense. But the complement of H is the union of these sets. Hence the complement is of first category. Now the second assertion follows from the first. □

Proof of Proposition 2.13. Let $(\mathfrak{G}, \mathfrak{s})$ be the completion of (G, σ). By Theorem 2.9, we can view $\mathfrak{G}$ is a metric group containing G as a dense subset. (The embedding of G into $\mathfrak{G}$ is isometric so G is homeomorphic to its image in $\mathfrak{G}$.) Since G is completely metrizable, it is a G_δ subset of $\mathfrak{G}$ by [Wil70, Theorem 24.12]. Hence Lemma 2.15 implies that $G = \mathfrak{G}$. This says that σ was complete to begin with. □

Corollary 2.16. *Let G and H be Polish groups and $\varphi : H \to G$ a continuous injection. Then $\varphi(H)$ is closed in G if and only if φ is a homeomorphism onto its range. More generally, if $\varphi : H \to G$ is a homeomorphism onto its range and H is Polish, then $\varphi(H)$ is closed (even if G is not).*

[8] To be precise, Moore defines a second countable group be Polish if σ_G is complete. His [Moo76, Proposition 1] is just the statement that he could have used the usual definition.

Proof. If $\varphi(H)$ is closed, then $\varphi(H)$ is Polish. Hence φ is a Borel isomorphism (see [Arv76, §3]), and φ^{-1} is a Borel homomorphism and therefore continuous [Wil07, Theorem D.11].

Now it suffices to prove the second assertion. If φ is a homeomorphism onto its range, then $\varphi(H)$ is complete with respect to the metric inherited from H. But Proposition 2.13, it is also complete in the symmetric metric $\sigma = \sigma_{\varphi(H)}$. Suppose $h_n \to g$ in G with each $h_n \in \varphi(H)$. Then $\{h_n\}$ is clearly Cauchy with respect to σ. Hence $g \in \varphi(H)$ and the latter is closed as required. $\square$

Remark 2.17. The second part of Corollary 2.16 asserts that Polish groups are what used to be called *absolutely closed* (see [Num80] for more on this).

Theorem 2.18. *Let*

$$e \longrightarrow H \xrightarrow{i} E \xrightarrow{j} G \longrightarrow e$$

be a short exact sequence of topological groups. Then E is Polish if and only if H and G are.

Proof. If E is Polish, then we can identify H with a closed subgroup of E. Hence H is Polish. It turns out that showing E/G is Polish is actually hard. Moore attributes it to a preprint of Brown's in [Moo76, Proposition 3]. Brown's pre-print referenced by Moore never made in into print as titled, but the result does seem to be implicit in [Bro72]. However, as near as I can tell, Brown merely refers to an old result of Hausdorff in [Hau34]. Sadly the paper is in German and no specific reference is given. I'm told it says that if $f : X \to Y$ is a continuous, open, surjection of a completely metrizable space onto a metrizable space, then the later is completely metrizable. (Of course, that suffices. But I haven't sorted the details out — far from it.) In [Moo76, Proposition 3], Moore claims that if E is abelian and if d is a complete metric on E, then one gets a complete metric on E/G via $\bar{d}(xH, yH) = d(xH, yH) = \inf\{\, d(x', y') : x' \in xH \text{ and } y' \in yH \,\}$.[9]

Now assume H and G are Polish. We know from Lemma 1.8 that E is second countable. Hence we can equip it with our symmetric metric and form its completion $\mathfrak{E}$ as above. Note that $\mathfrak{E}$ is Polish.

Let $k : E \to \mathfrak{E}$ be the natural injection. Note that $j : E \to G$ is uniformly continuous for the symmetric metrics: let V be a neighborhood of e in G. Let

[9] The only issue is completeness. It is not hard to see that $\bar{d}$ is always a peudo-metric on G/H — see [HR63, II.8.14] for the proof that the triangle inequality holds. To get a metric, it necessary to assume that convergence in d implies convergence in the right-invariant metric λ on G. (This is the case for us, as we can use the symmetric metric $\sigma = \rho + \lambda$.) Then if $\bar{d}(xH, H) = 0$, we also have $\lambda(xH, H) = \lambda(x, H) =$, and $x \in H$.

$D_V^G = \{\,(x,y) \in G \times G : x^{-1}y \in V \text{ and } xy^{-1} \in V\,\}$. Let $W = j^{-1}(V)$ and consider $D_W^E = \{\,(f,g) \in E \times E : f^{-1}g \in W \text{ and } fg^{-1} \in W\,\}$. Clearly if $(f,g) \in D_W^E$, then $\big(j(f), j(g)\big) \in D_V^G$. This shows j is uniformly continuous; hence j has an extension $\bar{j} : \mathfrak{E} \to G$.

Then we have a commutative diagram

$$\begin{tikzcd} & & \mathfrak{E} \arrow[dr, "\bar{j}"] & & \\ e \arrow[r] & H \arrow[ur, dotted, "i'"] \arrow[r, "i"] & E \arrow[u, "k"] \arrow[r, "j"] & G \arrow[r] & e, \end{tikzcd}$$

where $i' := k \circ i$.

Let $\mathfrak{H} = \ker \bar{j}$. Suppose that $\mathfrak{h} = [g_n] \in \ker \bar{j} = \mathfrak{H}$. Then $j(g_n) \to e_G$ in G. Since j is open, after passing to a subsequence and relabeling, there are $k_n \in E$ such that $k_n \to e_E$ in E and $j(k_n) = j(g_n)$. Then each $k_n^{-1}g_n \in i(H)$ and $k\big(k_n^{-1}g_n\big) \to \mathfrak{h}$ in $\mathfrak{E}$. Thus $i'(H)$ is dense in $\mathfrak{H}$. But i' is a homeomorphism onto its range (since both i and k are). Therefore $i'(H)$ is closed in $\mathfrak{E}$ by Corollary 2.16. Hence $i'(H) = \mathfrak{H}$. Now if $\mathfrak{g} \in \mathfrak{E}$, there is a $g \in E$ such that $j(g) = \bar{j}(\mathfrak{g})$. Thus $k(g)i'(h) = \mathfrak{g}$ for some $h \in H$. But then $k(gh) = \mathfrak{g}$ and k is onto. Thus E is complete. $\square$

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