A (Very) Short Course on C^*-Algebras

Dana P. Williams

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ABSTRACT: These are a minimally polished set of lecture notes for a course on C^*-algebras which I've given several times at Dartmouth College. The LaTeX version here was produced during the Spring 2011 instance of the course. I used Murphy's book [Mur90] as the primary source and no doubt some of the arguments here are his.

It seem unlikely that these notes will ever be published. In addition to Murphy's book, there are the classics by Dixmier [Dix77, Dix81], Takesaki [Tak79] and Pedersen [Ped82], and a host of newer books such as Davidson [Dav96] and Kadison-Ringrose [KR83]. There is even an Encyclopaedia of Mathematical Sciences volume by Blackadar [Bla06].

Nevertheless, I am providing these to future graduate students as a quick introduction. Anyone if free to use them as they see fit. However, I would be very grateful to hear about typos, corrections and especially suggestions to improve the exposition. In particular, I will continue to tinker with them so the page and theorem numbers will certainly change over time.

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Department of Mathematics
Dartmouth College
Hanover, NH 03755-3551 USA
`dana.williams@dartmouth.edu`

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Nothing yet.

Chapter 1

Algebras

1.1 Getting Started

An *algebra* is a vector space A whose underlying additive abelain group has a ring structure compatible with the given scalar multiplication:

$$\lambda(ab) = (\lambda a)b = a(\lambda b) \quad \text{for all } a, b \in A \text{ and } \lambda \in \mathbf{C}.$$

Unless we have good reason, and that will be rarely if at all, our algebras are always vector spaces over the complex numbers. A good example to keep in mind is the algebra $n \times n$ matrices, $M_n := M_n(\mathbf{C})$, with the usual matrix operations from linear algebra. An algebra is called commutative if $ab = ba$ for all a, b.

A *normed vector space* is a vector space A equipped with a norm usually denoted by $\|\cdot\|$. Recall that a *norm* is just a function $\|\cdot\| : A \to [0, \infty)$ such that

N1: (definiteness) $\|a\| = 0$ if and only if $a = 0$,

N2: (homogeneity) $\|\lambda a\| = |\lambda|\|a\|$, for all $a \in A$ and $\lambda \in \mathbf{C}$ and

N3: (triangle inequality) $\|a + b\| \leq \|a\| + \|b\|$ for all $a, b \in A$.

If $\|\cdot\|$ satisfies only $N2$ and $N3$, then it is called a *seminorm*.

An algebra A equipped with a norm is called a *normed algebra* if the norm is *submultiplicative*; that is, if

$$\|ab\| \leq \|a\|\|b\| \quad \text{for all } a, b \in A.$$

An algebra is called *unital* if A has a ring identity $\mathbf{1}_A$. In unital normed algebras we always assume that $\|\mathbf{1}_A\| = 1$.

Of course, if A is a normed algebra, then the norm induces a metric on A which in turn induces a topology on A called the *norm topology*.

Lemma 1.1. *If A is a normed algebra, then all the algebraic operations are continuous in the norm topology on A.*

Proof. For example, from the submultiplicativity of the norm implies that

$$\begin{aligned}\|xy - ab\| &\leq \|xy - xb\| + \|xb - ab\| \\ &\leq \|x\|\|y - b\| + \|x - a\|\|b\|,\end{aligned}$$

and it easily follows that multiplication is continuous from $A \times A$ to A. Addition and scalar multiplication are even easier. □

Definition 1.2. A normed algebra which is complete in the metric induced by the norm is a called a *Banach algebra*.

Example 1.3. Let S be a set and let $\ell^\infty(S)$ be the collection of all bounded complex-valued functions on S. Then $\ell^\infty(S)$ is a unital Banach algebra with respect to the usual pointwise operations:

$$\begin{aligned}(f+g)(s) &:= f(s) + g(s) & \qquad (fg)(s) &:= f(s)g(s) \\ (\lambda f)(s) &:= \lambda f(s) & \mathbf{1}_{\ell^\infty(S)}(s) &:= 1\end{aligned}$$

for all $s \in S$. The norm is given by

$$\|f\| := \|f\|_\infty = \sup_{s \in S} |f(s)|.$$

Of course, $\ell^\infty(S)$ is an example of a commutative Banach algebra.

Remark 1.4. It is important to keep in mind that every normed vector space V can be viewed as a dense subspace of a Banach space $\overline{V}$ — called the *completion* of V. In fact, the natural map $\iota : V \to V^{**}$ is isometric. Since X^{**} is complete, the closure $\overline{\iota(V)}$ is a Banach space containing an isometric copy of V.

Definition 1.5. If A is an algebra, then a vector subspace B is called a *subalgebra* if $b, c \in B$ implies that $bc \in B$.

Notice that a subalgebra is itself an algebra. A subalgebra of a normed algebra is a normed algebra. The closure of a subalgebra of a normed algebra is a normed algebra. Therefore the closure of any subalgebra of a Banach algebra is again a Banach algebra.

Example 1.6. Suppose that X is a topological space. Let $C^b(X)$ be the collection of bounded continuous functions on X. Then $C^b(X)$ is a closed subalgebra of $\ell^\infty(X)$. Hence, $C^b(X)$ is a Banach algebra.

Without any hypotheses on X, $C^b(X)$ could be reduced to the constant functions. But if X is locally compact and Hausdorff, then $C^b(X)$ contains lots of functions by Urysohn's Lemma (and its variations for general locally compact spaces — see [Wil07, §1.2.2]). However, if X is locally compact, but not compact, then $C^b(X)$ will likely be nonseparable and therefore pathological. (Consider the bounded continuous functions on the real line.) To keep things under control, we introduce the following concept.

Definition 1.7. If X is a locally compact Hausdorff space, then a continuous function f on X is said to *vanish at infinity* if

$$\{\, x \in X : |f(x)| \geq \epsilon \,\}$$

is compact for all $\epsilon > 0$. The collection of all such f is denoted by $C_0(X)$.

It is practically immediate that $C_0(X)$ is a subalgebra of $C^b(X)$. With a bit more work, you can show that $C_0(X)$ is closed in $C^b(X)$. Therefore, $C_0(X)$ is a Banach algebra.

If we write $C(X)$ for the vector space of all continuous functions on X, then the *support* of $f \in C(X)$ is

$$\operatorname{supp} f := \overline{\{\, x \in X : f(x) \neq 0 \,\}}.$$

Note that by definition, the support of a function is always closed. Of course, we say that f has compact support if $\operatorname{supp} f$ is compact. We write $C_c(X)$ for the collection of compactly supported continuous functions on X. It is not hard to see that $C_c(X)$ is a dense subalgebra of $C_0(X)$.

The Banach algebras $C_0(X)$ (for X locally compact Hausdorff) will be our favorite example of a *commutative* Banach algebra. To get interesting noncommutative examples, a natural path is via operators. Recall that if V and W are normed vector spaces, then a linear map or operator $T : V \to W$ is called *bounded* if there is a $k \geq 0$ such that

$$\|T(v)\| \leq k\|v\| \quad \text{for all } v \in V.$$

A bounded operator is continuous for the respective norm topologies on V and W. Conversely, if $T : V \to W$ is linear, then continuity at just one point in V implies that T is bounded. We let $B(V, W)$ be the collection of all bounded operators from V to W and write $B(V)$ in place of $B(V, V)$. We are especially interested in $B(V)$ as it becomes an algebra with respect to composition

$$ST(v) := S\big(T(v)\big).$$

The *operator norm* on $B(V)$ is given by

$$\|T\| := \sup_{v \neq 0} \frac{\|T(v)\|}{\|v\|}.$$

We're supposed to recall the following facts about the operator norm from our functional analysis classes.

Lemma 1.8. *Let V be a normed vector space.*

(a) *The operator norm is a norm.*
(b) *If $T \in B(V)$, then $\|T(v)\| \leq \|T\|\|v\|$ for all $v \in V$.*
(c) *If $T, S \in B(V)$, then $\|TS\| \leq \|T\|\|S\|$.*
(d) *If $T \in B(V)$, then $\|T\| = \sup_{\|v\|=1} \|T(v)\|$.*
(e) *If $T \in B(V)$, then $\|T\| = \inf\{ k \geq 0 : \|T(v)\| \leq k\|v\| \text{ for all } v \in V \}$.*

In particular, if V is a normed vector space, then $B(V)$ is a normed algebra with respect to the operator norm.

Proposition 1.9. *If X is a Banach space, then $B(X)$ is a unital Banach algebra.*

Proof. Since the identity operator is an identity, we just need to see that $B(X)$ is complete. Let $\{T_n\}$ be a Cauchy sequence in $B(X)$. Fix $x \in X$. Since

$$\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\|\|x\|,$$

it follows that $\{T_n(x)\}$ is a Cauchy sequence in X. Since X is complete by assumption, there is an element, $T(x)$, such that

$$T(x) = \lim_{n \to \infty} T_n(x) \quad \text{for all } x \in X.$$

This clearly defines a linear operator $T : X \to X$. Since $\{\|T_n\|\}$ is easily seen to be bounded and since

$$\|T(x)\| = \lim_n \|T_n(x)\| \leq \limsup_n \|T_n\|\|x\|,$$

it follows that T is bounded.

To complete the proof, we must show that $T_n \to T$ in $B(X)$. That is, given $\epsilon > 0$, we need to produce an N such that $n \geq N$ implies $\|T_n - T\| < \epsilon$. But we can choose

N such that $n, m \geq N$ implies $\|T_n - T_m\| < \epsilon/2$. Now if $x \in X$ has norm one and if $n \geq N$, then

$$\begin{aligned} \|(T_n - T)(x)\| &= \lim_k \|T_n(x) - T_k(x)\| \\ &\leq \limsup_k \|T_n - T_k\| \|x\| \\ &< \epsilon \end{aligned}$$

as required. □

Example 1.10. We can identify the complex $n \times n$ matrices M_n with $B(\mathbf{C}^n)$ in the usual way. Then with the operator norm, M_n is a Banach algebra.

The 2×2 complex matrices are a nice simple model of a noncommutative Banach algebra. But the subtlety of the operator norm, even for M_2, should not be underestimated. For example, it takes a little work to see that

$$\left\| \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \right\| = 2 + \sqrt{2}. \tag{1.1}$$

Suppose that B_λ is a subalgebra of A for all $\lambda \in \Lambda$. Then

$$B := \bigcap_{\lambda \in \Lambda} B_\lambda$$

is a subalgebra of A. It follows that given any subset $S \subset A$, there is a smallest subalgebra $\operatorname{alg}(S)$ containing S.

To illustrate the above construction, we want to remind ourselves of one of the very basic tools for understanding function spaces such as $C(X)$, for X compact Hausdorff; namely, the Stone-Weierstrass Theorem.

Theorem 1.11 (Stone-Weierstrass). *Suppose that X is a compact Hausdorff space. Let A be a closed subalgebra of $C(X)$ that separates points and is closed under complex conjugation. Then either A is all of $C(X)$ or $A = \{ f \in C(X) : f(x_0) = 0 \}$ for some point $x_0 \in X$.*

The version of the theorem stated here can be found in [Fol99, Theorem 4.51] or [Con85, Theorem 8.1 and Corollary 8.2] or [Ped89, Theorem 4.3.4 and Corollary 4.3.5]. As a neat application, we have the following example.

Example 1.12. Let $\mathbf{T} := \{ z \in \mathbf{C} : |z| = 1 \}$. Let f be the identity function on $\mathbf{T}$: $f(z) = z$ for all $z \in \mathbf{T}$. Then the Stone-Weierstrass Theorem implies that $\operatorname{alg}(\{f, \bar{f}\})$ is dense in $C(\mathbf{T})$.

Example 1.13. Getting our hands on $\mathrm{alg}(\{f, \bar{f}\})$ is a bit more tricky. Recall that the algebra $\mathbf{C}[x,y]$ of *polynomials in two variables* can be defined as the vector space with basis $\{ x^n y^m : n \text{ and } m \text{ are nonnegative integers} \}$ and ring structure compatible with $(x^n y^m)(x^k y^l) = x^{n+k} y^{m+l}$. If a and b are elements of an algebra A which commute — that is, such that $ab = ba$ — the there is an algebra homomorphism of $\mathbf{C}[x,y]$ into A taking $p = \sum_{n,m} \lambda_{n,m} x^n y^m$ in $\mathbf{C}[x,y]$ to $p(a,b) = \sum_{n,m} \lambda_{n,m} a^n b^m$ in A. The image of this homomorphism is clearly $\mathrm{alg}\,\{a,b\}$.[1] Therefore $\mathrm{alg}\,\{f, \bar{f}\}$ consists of functions g of the form $g(z) = q(z, \bar{z})$ for $q \in \mathbf{C}[x,y]$.

Definition 1.14. A vector subspace I of an algebra A is called an *left ideal* of A if $a \in A$ and $b \in I$ implies $ab \in I$. A *right ideal* is defined similarly, and we call I an *ideal* if it is both a right and left ideal.

Just as for $\mathrm{alg}(S)$, there is a smallest ideal of A, $I(S)$, containing a subset $S \subset A$.

It is routine to check that if I is an ideal in an algebra A, then the quotient space A/I becomes an algebra with respect to the multiplication

$$(a + I)(b + I) := ab + I.$$

Let $q : A \to A/I$ be the *quotient map*. Assuming A is a normed algebra, then *quotient norm* on A/I is given by

$$\|q(a)\| := \inf_{y \in I} \|a + y\|.$$

Theorem 1.15. *Suppose that A is a normed algebra and that I is an ideal in A.*

(a) *The quotient norm is a submultiplicative seminorm on A/I.*

(b) *The quotient norm is a norm exactly when I is closed in A.*

(c) *If I is a proper closed ideal, then the quotient map $q : A \to A/I$ has norm 1.*

(d) *If A is a Banach algebra and if I is closed, then A/I is a Banach algebra (with respect to the quotient norm).*

Proof. Items (a)–(c) are routine and are valid when I is merely a subspace of A. (For example, see [Rud87, §18.15] or [Fol99, Exercises 12 & 14 in §5.1].)

Since it is also straightforward to check that A/I is an algebra when I is an ideal, we'll only show that the quotient norm is submultiplicative here. To this end, fix $a, b \in A$ and $\epsilon > 0$. Let $y, z \in I$ be such that $\|q(a)\| + \epsilon > \|a + y\|$ and

[1] The situation is considerably more complicated if a and b don't commute.

$\|q(b)\| + \epsilon > \|b+z\|$. Then

$$\begin{aligned}\big(\|q(a)\|+\epsilon\big)\big(\|q(b)\|+\epsilon\big) &> \|a+y\|\|b+z\| \\ &\geq \|(a+y)(b+z)\| \\ &= \|ab+by+az+yz\| \\ &\geq \|q(ab)\|.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this suffices. □

Not surprisingly, commutative Banach algebras are much easier to study. In spite of the fact that we are most interested in algebras modeling $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ — which are most emphatically not commutative — the theory of commutative Banach algebras plays a very important role in the sequel. Hence it is nice to see that there are some quite sophisticated examples easily at hand.

Example 1.16. Consider the Banach space $\ell^1(\mathbf{Z})$.[2] If f and g are in $\ell^1(\mathbf{Z})$, then their *convolution*, $f * g$ is defined by

$$f * g(n) = \sum_{m=-\infty}^{\infty} f(m)g(n-m).$$

It is a straightforward exercise (see E 1.3.6(a)) to see that $f * g$ defines an element of $\ell^1(\mathbf{Z})$ and that $\ell^1(\mathbf{Z})$ is a commutative Banach algebra under convolution.

Example 1.17 (The Disk Algebra). Let $D = \{\, z \in \mathbf{C} : |z| < 1 \,\}$ be the open unit disk. We'll naturally write $\overline{D}$ for its closure $\{\, z \in \mathbf{C} : |z| \leq 1 \,\}$, and $\mathbf{T}$ for its boundary. Consider the subalgebra B of $C(\overline{D})$ of functions $g \in C(\overline{D})$ such that $g|_D$ is holomorphic. By Morera's Theorem, the uniform limit of holomorphic functions is holomorphic, and it follows B is a closed subalgebra of $C(\overline{D})$. Let $A(D)$ be the image in $C(\mathbf{T})$ of B via the restriction map. Since the map $g \mapsto g|_{\mathbf{T}}$ is isometric by the Maximum modulus principle, $A(D)$ is complete and therefore closed in $C(\mathbf{T})$. Hence $A(D)$ is a closed subalgebra of $C(\mathbf{T})$ and is a Banach algebra in its own right usually called the *disk algebra*. It consists of those functions in $C(\mathbf{T})$ that have holomorphic extensions to D. As described above, B and $A(D)$ are isometrically isomorphic and some treatments even call B the disk algebra.

[2] Recall that $\ell^1(\mathbf{Z}) = L^1(\mathbf{Z}, \nu)$, where ν is counting measure, is the set of functions $f : \mathbf{Z} \to \mathbf{C}$ such that $\lim_{N\to\infty} \sum_{n=-N}^{n=N} |f(n)| < \infty$.

Exercises

E 1.1.1. If V is a normed vector space, then we could define a *completion* of V to be a Banach space X together with an isometric injection j of V onto a dense subspace of X. Remark 1.4 on page 2 shows that every normed vector space has a completion. Show that if (X, j) and (X', j') are completions of V, then there is an isometric isomorphism $\psi : X \to X'$ such that $j' = \psi \circ j$. (This justifies the use of the definite article in referring to *the* completion of V.) Note that $A(D)$ is isometrically isomorphic to the subalgebra of $C(\overline{D})$ consisting of those $f \in C(\overline{D})$ holomorphic on D.

E 1.1.2. Suppose that X is a normed vector space. Then X is a Banach space (that is, X is complete) if and only if every absolutely convergent series in X is convergent.

E 1.1.3. Let X be a normed vector space and suppose that S and T are bounded linear operators on X. Show that $\|ST\| \le \|S\|\|T\|$.

E 1.1.4. Suppose that V and W are normed vector spaces and that $T : V \to W$ is linear. Show that T is continuous if and only if T is bounded.

E 1.1.5. Provide a proof of parts (a)–(c) of Theorem 1.15.

E 1.1.6. Let X be a locally compact Hausdorff space. Show that $C_0(X)$ is a closed subalgebra of $C^b(X)$, and that $C_c(X)$ is dense in $C_0(X)$.

E 1.1.7. Show that $C^b(\mathbf{R})$ is not a separable Banach space, but that $C_0(\mathbf{R})$ is separable.

E 1.1.8. Suppose that X is a compact Hausdorff space. If E is a closed subset of X, define $I(E)$ to be the ideal in $C(X)$ of functions which vanish on E.

(a) Let J be a closed ideal in $C(X)$ and let $E = \{\, x \in X : f(x) = 0 \text{ for all } f \in J \,\}$. Prove that if U is an open neighborhood of E in X, then there is a $f \in J$ such that $0 \le f(x) \le 1$ for all x and such that $f(x) = 1$ for all x in the compact set $X \setminus U$.

(b) Conclude that $J = I(E)$ in part (a), and hence, conclude that *every closed* ideal in $C(X)$ has the form $I(E)$ for some closed subset E of X.

—Answer on page 131

E 1.1.9. Suppose that X is a (non-compact) locally compact Hausdorff space. Let X^+ be the *one-point compactification* of X (also called the Alexandroff compactification: see [Kelly; Theorem 5.21] or [Folland, Proposition 4.36]). Recall that $X^+ = X \cup \{\, \infty \,\}$ with $U \subseteq X^+$ open if and only if either U is an open subset of X or $X^+ \setminus U$ is a *compact* subset of X.

(a) Show that $f \in C(X)$ belongs to $C_0(X)$ if and only if the extension

$$\tilde{f}(\tilde{x}) = \begin{cases} f(\tilde{x}) & \text{if } \tilde{x} \in X, \text{ and} \\ 0 & \text{if } \tilde{x} = \infty. \end{cases}$$

is continuous on X^+.

(b) Conclude that $C_0(X)$ can be identified with the maximal ideal of $C(X^+)$ consisting of functions which 'vanish at ∞.'

—Answer on page 131

E 1.1.10. Use the above to establish the following ideal theorem for $C_0(X)$.

Theorem: Suppose that X is a locally compact Hausdorff space. Then every closed ideal J in $C_0(X)$ is of the form

$$J = \{\, f \in C_0(X) : f(x) = 0 \text{ for all } x \in E \,\}$$

for some closed subset E of X.

—Answer on page 132

E 1.1.11. Use one-point compactifications to generalize the Stone-Weierstrass Theorem on page 5 as follows. Suppose that X is a locally compact Hausdorff space and that A is a closed subalgebra of $C_0(X)$ that separates points and is closed under complex conjugation. Then either $A = C_0(X)$ or there is an $x_0 \in X$ such that $A = \{\, f \in C_0(X) : f(x_0) = 0 \,\}$.

—Answer on page 132

1.2 The Spectrum

Definition 1.18. If A is a unital Banach algebra, then we let $\mathrm{Inv}(A)$ be the group of invertible elements in A.

Proposition 1.19. *Suppose that A is a unital Banach algebra and that $a \in A$ satisfies $\|a\| < 1$. Then $\mathbf{1}_A - a \in \mathrm{Inv}(A)$ and*

$$(\mathbf{1}_A - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Proof. Since $\|a\| < 1$, $\sum_{n=0}^{\infty} a^n$ is absolutely convergent. Hence the series converges to an element $b \in A$ (see E 1.1.1.1.2). Thus if $b_N := \sum_{n=1}^{N} a^n$, then $b_N \to b$ in A. But

$$(\mathbf{1}_A - a)b_N = b_N(\mathbf{1}_A - a) = \mathbf{1}_A - a^{N+1}.$$

Taking limits, we have $(\mathbf{1}_A - a)b = \mathbf{1}_A = b(\mathbf{1}_A - a)$ as required. □

Corollary 1.20. *If A is a unital Banach algebra, then $\mathrm{Inv}(A)$ is open.*

Proof. Let $a \in \mathrm{Inv}(A)$. Then if $h \in A$, we have

$$a - h = a(\mathbf{1}_A - a^{-1}h).$$

Therefore if $\|h\| < \|a^{-1}\|^{-1}$, then $\|a^{-1}h\| < 1$ and $a - h \in \mathrm{Inv}(A)$ by the previous result. □

Definition 1.21. If A is a unital Banach algebra, then the *spectrum* of $a \in A$ is

$$\sigma(a) := \{\, \lambda \in \mathbf{C} : \lambda \mathbf{1}_A - a \notin \mathrm{Inv}(A) \,\}.$$

The *spectral radius* of a is

$$\rho(a) := \sup\{\, |\lambda| : \lambda \in \sigma(a) \,\}.$$

Example 1.22. If A is the algebra M_n of $n \times n$ complex matrices, then the spectrum of $T \in M_n$ is the collection of eigenvalues of T. Of course, then the spectral radius of T is just the largest modulus of any eigenvalue. For example, if $T \in M_2$ is the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, then $\sigma(T) = \{0\}$ and $\rho(T) = 0$.

Example 1.23. If $f \in A := C\big([0,1]\big)$, then the spectrum of f coincides with its range, $f\big([0,1]\big)$, and the spectral radius of f is its norm, $\|f\|_\infty$.

As we will see in the next result, it is not hard to see that the spectrum is a compact subset of the plane. However, it requires a bit of work to see that in a Banach algebra, the spectrum of any element is always nonempty. This result requires completeness. For example, the field of quotients, $\mathbf{C}(z)$, of complex polynomials $\mathbf{C}[z]$ in one variable, is a complex algebra in an obvious way. But if $p \in \mathbf{C}(z)$ is the identity function $p(z) = z$ for all z, then $\sigma(p) = \emptyset$.

The next result is the heart of the Gelfand theory to be developed in the next section. I like the proof provided here because I enjoy being able to invoke some of the nice theorems from complex analysis we work so hard to prove in our introductory courses. But the proof requires two bits of sophistication that might bother some.[3]

Firstly, we work with what are often called *strongly* holomorphic functions $F : \Omega \to A$ from an open connected subset $\Omega \subset \mathbf{C}$ taking values in a complex Banach algebra A. This just means that for each $z \in \Omega$, the limit

$$f'(z) := \lim_{h\to 0} \frac{1}{h}(f(z+h) - f(z))$$

exists in A. Notice that if $\varphi \in A^*$, then $\varphi \circ f$ is an ordinary holomorphic function on Ω. In particular, if $\Omega = \mathbf{C}$ and f were bounded, then $\varphi \circ f$ would be a bounded entire function and constant by Liouville's Theorem. Since A^* separates points in A, this forces f itself to be constant. In this way, we can apply standard results from complex analysis to strongly holomorphic functions. I'll do this without much comment in the sequel.

A second bit of apparent legerdemain are A-valued integrals of the form

$$a = \int_c^d f(t)\,dt \tag{1.2}$$

where $f : [c, d] \to A$ is a continuous function. (In fact, the integrals that appear will be contour integrals, but contour integrals reduce to integrals of the form (1.2).) Of course, there are vast references for Banach space valued integrals. But here one can make do my mimicking the Riemann theory. (For a more sophisticated treatment, consider [Wil07, §1.5] or even [Wil07, Appendix B.1].) Once you are convinced that there is an $a \in A$ which is the value of the integral, you can return to the scalar case by noting that (1.2) amounts to

$$\varphi(a) = \int_c^d \varphi\big(f(t)\big)\,dt \quad \text{for all } \varphi \in A^*. \tag{1.3}$$

[3] If the prerequisites below are a bit daunting, Arveson has a nice approach in [Arv02, §§1.6–7] which is very elegant and seems to involve less overhead.

Note that (1.3) is actually a statement about the (Riemann) integral of a continuous $\mathbf{C}$-valued function. The sophisticated bit is the assertion that there is a single $a \in A$ that works for all φ.

Theorem 1.24. *Suppose that A is a unital Banach algebra and that $a \in A$.*

(a) *(Gelfand) $\sigma(a)$ is compact and* nonempty.

(b) *(Beurling) $\rho(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|a^n\|^{\frac{1}{n}}$.*

Remark 1.25. It is worth mentioning that the existence of the limit in part (b) is highly nontrivial and it is part of the statement of the theorem that the limit always exists (and equals $\rho(a)$).

Proof. If $|\lambda| > \|a\|$, then $\lambda\mathbf{1}_A - a = \lambda(\mathbf{1}_A - \lambda^{-1}a) \in \operatorname{Inv}(A)$. Therefore, $\rho(a) \leq \|a\|$; in particular $\sigma(a)$ is bounded.

Define $g : \mathbf{C} \to A$ by $g(\lambda) = \lambda\mathbf{1}_A - a$. Then g is continuous. Thus,

$$\Omega = \{\, \lambda \in \mathbf{C} : \lambda \notin \sigma(a) \,\} = g^{-1}\big(\operatorname{Inv}(A)\big)$$

is open. Therefore, $\sigma(a)$ is closed. Of course, a closed and bounded subset of $\mathbf{C}$ is compact. Note that if $a = 0$, then $\sigma(a) = \{0\}$ and both parts of the theorem are clear. Hence, we'll assume $a \neq 0$ for the rest of the proof.

Now define $f : \Omega \to \operatorname{Inv}(A)$ by $f(\lambda) = (\lambda\mathbf{1}_A - a)^{-1}$.[4] To see that f is strongly holomorphic on Ω, consider

$$\begin{aligned}\lim_{h\to 0} \frac{1}{h}\big(f(\lambda+h) - f(\lambda)\big) &= \lim_{h\to 0}\frac{1}{h}\big((\lambda\mathbf{1}_A - a + h\mathbf{1}_A)^{-1} - (\lambda\mathbf{1}_A - a)^{-1}\big)\\ &= \lim_{h\to 0}\frac{1}{h}\Big((\mathbf{1}_A + f(\lambda)h)^{-1} - \mathbf{1}_A\Big)f(\lambda),\end{aligned}$$

Now if h is so small that $\|f(\lambda)h\| < 1$, then

$$= \lim_{h\to 0}\frac{1}{h}\Big(\sum_{n=1}^{\infty}\big(-f(\lambda)h\big)^n\Big)f(\lambda) = -f(\lambda)^2.$$

We have shown that $\lambda \mapsto f(\lambda)$ is a strongly holomorphic A-valued function on Ω.

[4]One often writes $f(\lambda) = R(x,\lambda)$ and refers to $f(\lambda)$ as the resolvent of a at λ.

Now if $\lambda > \|a\|$, then

$$\begin{aligned} f(\lambda) = (\lambda \mathbf{1}_A - a)^{-1} &= \lambda^{-1}\Big(\mathbf{1}_A - \frac{a}{\lambda}\Big)^{-1} \\ &= \frac{1}{\lambda}\sum_{n=0}^{\infty}\Big(\frac{a}{\lambda}\Big)^n \\ &= \frac{1}{\lambda}\mathbf{1}_A + \frac{1}{\lambda^2}a + \cdots . \end{aligned}$$

And the convergence is uniform on circles Γ_r centered at 0 provided $r > \|a\|$. Since $\int_{\Gamma_r} \lambda^n \, d\lambda$ is $2\pi i$ if n is -1 and 0 otherwise, we have

$$a^n = \frac{1}{2\pi i}\int_{\Gamma_r} \lambda^n f(\lambda)\, d\lambda \qquad n = 0, 1, 2, \dots, \tag{1.4}$$

where we have discussed how to interpret the A-valued integral prior to the statement of the theorem.

Since $|\lambda| > \rho(a)$ implies that $\lambda \in \Omega$, it follows that if $r, r' > \rho(x)$, then Γ_r and $\Gamma_{r'}$ are *homotopic* in Ω! Thus, Equation 1.4 holds for all $r > \rho(x)$.

Now if $\Omega = \mathbf{C}$, then f is entire and bounded (since $|\lambda| > 2\|a\|$ implies that $\|f(\lambda)\| \le \frac{1}{|\lambda|}\big(\frac{1}{1-\frac{\|a\|}{|\lambda|}}\big) \le \frac{1}{|\lambda|-\|a\|} \le \frac{1}{\|a\|}$). Therefore, f would be constant. It would then follow from Equation 1.4, with $n = 0$, that $\mathbf{1}_A = 0$; this is a contradiction, so $\sigma(a) \neq \emptyset$. We've proved (a).

Now let $M(r) = \max_{\lambda \in \Gamma_r} \|f(\lambda)\|$. Using Equation 1.4,

$$\|a^n\| \le r^{n+1} M(r).$$

Thus

$$\limsup_n \|a^n\|^{\frac{1}{n}} \le r,$$

and it follows that

$$\limsup_n \|a^n\|^{\frac{1}{n}} \le \rho(a).$$

Now, if $\lambda \in \sigma(a)$, then

$$\begin{aligned} (\lambda^n \mathbf{1}_A - a^n) &= (\lambda \mathbf{1}_A - a)(\lambda^{n-1}\mathbf{1}_A + \lambda^{n-2}a + \cdots + a^{n-1}) \\ &= (\lambda^{n-1}\mathbf{1}_A + \lambda^{n-2}a + \cdots + a^{n-1})(\lambda \mathbf{1}_A - a) \end{aligned}$$

implies that $\lambda^n \in \sigma(a^n)$ — otherwise, $(\lambda^{n-1}\mathbf{1}_A + \cdots + a^{n-1})(\lambda^n \mathbf{1}_A - a^n)^{-1}$ is an inverse for $(\lambda \mathbf{1}_A - a)$. As a consequence, if $\lambda \in \sigma(a)$, then $|\lambda^n| \leq \rho(a^n) \leq \|a^n\|$. In particular, $|\lambda| \leq \|a^n\|^{\frac{1}{n}}$. Therefore, $\rho(a) \leq \inf_{n\geq 1} \|a^n\|^{\frac{1}{n}}$. Combining with the previous paragraph,

$$\limsup_n \|a^n\|^{\frac{1}{n}} \leq \rho(a) \leq \inf_{n\geq 1} \|a^n\|^{\frac{1}{n}} \leq \liminf_n \|a^n\|^{\frac{1}{n}}.$$

This completes the proof. □

Corollary 1.26. *For all a in a unital Banach algebra A, we have $\rho(a) \leq \|a\|$.*

The next result, often called the Gelfand-Mazur Theorem, is an deep result that follows easily from the fact that the spectrum of any element must be nonempty.

Corollary 1.27. *A unital Banach algebra in which every nonzero element is invertible (that is, a division ring) is isometrically isomorphic to* $\mathbf{C}$.

Proof. Suppose that A is a Banach algebra that is also a division ring. If λ and μ are distinct complex numbers, then at most one of $\lambda\mathbf{1} - a$ and $\mu\mathbf{1} - a$ can be zero. Therefore the spectrum of any $a \in A$ is exactly one point; say, $\sigma(a) = \{h(a)\}$. It is not hard to see that $h : A \to \mathbf{C}$ is the required map. For example, $h(a)\mathbf{1} - a = 0$ implies that $|h(a)| = \|a\|$. On the other hand,

$$\begin{aligned} h(a)h(b)\mathbf{1} - ab &= h(a)h(b)\mathbf{1} - h(a)b + h(a)b - ab \\ &= h(a)(h(b)\mathbf{1} - b) + (h(a)\mathbf{1} - a)b = 0 \end{aligned}$$

implies that $h(ab) = h(a)h(b)$. Showing that h is linear is easier. □

To Do: Need exercises. We can use the disk algebra to demonstrate spectral "impermanence" ahead of the spectral permanence result.

Exercises

E 1.2.1. Let A be a unital Banach algebra. Show that $x \mapsto x^{-1}$ is continuous from $\mathrm{Inv}(A)$ to $\mathrm{Inv}(A)$. (Hint: $(a-h)^{-1} - a^{-1} = ((1 - a^{-1}h)^{-1} - 1)a^{-1}$.)
—Answer on page 132

1.3 The Gelfand Transform

Definition 1.28. Suppose that A is a *commutative* Banach algebra. Then we write Δ, or $\Delta(A)$ if ambiguity exists, for the collection of *nonzero* complex homomorphisms $h : A \to \mathbf{C}$.

It is notable that we are not making any assumptions on the $h \in \Delta(A)$ other than that they are algebra homomorphisms. In particular, we have not made any continuity assumptions. As the next result shows, they are automatically continuous — at least on unital algebras. In fact, it will follow from Remark 1.35 that all complex homomorphisms are automatically continuous.

Theorem 1.29. *Suppose that A is a commutative* unital *Banach algebra.*

(a) Δ *is nonempty.*
(b) J *is a maximal ideal in A if and only if $J = \ker h$ for some $h \in \Delta$.*
(c) $\|h\| = 1$ *for all $h \in \Delta$.*
(d) *For each $a \in A$, we have $\sigma(a) = \{\, h(a) : h \in \Delta \,\}$.*

Before starting the proof, the following lemma will be useful.

Lemma 1.30. *If A is a unital Banach algebra, then every proper ideal is contained in maximal ideal and every maximal ideal is closed.*

Proof. If J is a proper ideal in A, then $J \cap \operatorname{Inv}(A) = \emptyset$. Since $\operatorname{Inv}(A)$ is open (by Corollary 1.20), we also have $\overline{J} \cap \operatorname{Inv}(A) = \emptyset$. Hence $\overline{J}$ is also proper. Therefore, all maximal ideals must be closed.

Now suppose again that J is a proper ideal and that $\{I_\lambda\}_{\lambda\in\Lambda}$ is a family of proper ideals containing J which is *totally ordered* by containment. Then their union,

$$I := \bigcup_{\lambda\in\Lambda} I_\lambda$$

must be disjoint from $\operatorname{Inv}(A)$. Thus I is a proper ideal majorizing all the I_λ. Hence Zorn's Lemma implies that existence of a maximal ideal containing J as required. □

Proof of Theorem 1.29. There must be at least one maximal ideal J by Lemma 1.30. Furthermore, if J is a maximal ideal, it is closed and the quotient map $q : A \to A/J$ has norm one by Theorem 1.15(c). Suppose that $a \in A$ is such that $q(a) \neq 0$. Let

$$M := \{\, ba + y : b \in A \text{ and } y \in J \,\}.$$

Then M is an ideal in A (since A is commutative) properly containing J. Since J is maximal, $M = A$ and there is a $b \in A$ and a $y \in J$ such that

$$\mathbf{1}_A = ba + y.$$

But then

$$q(b)q(a) = q(\mathbf{1}_A) = \mathbf{1}_{A/J},$$

and $q(a)$ is invertible in A/J. Thus A/J is a division ring and Corollary 1.27 implies that there is an isometric isomorphism $\varphi : A/J \to \mathbf{C}$. Then $h := \varphi \circ q$ is a complex homomorphism (of norm 1) with $\ker h = J$.

We have proved (a), and half of (b).

Suppose that $h \in \Delta$. We want to see that $\ker h$ is a maximal ideal. Suppose that $\ker h \subsetneq J$ for some ideal J. Then there is a $b \in J$ such that $h(b) \neq 0$. Then if $a \in A$,

$$a - h(a)h(b)^{-1}b \in \ker h \subset J.$$

This means that $a \in J$. Since $a \in A$ was arbitrary, $J = A$, and $\ker h$ is maximal. We have proved the rest of (b).

Note that (c) follows from (b) and the first part of the proof.

If $a \in \operatorname{Inv}(A)$, then for any $h \in \Delta$, we have $h(a)h(a^{-1}) = h(\mathbf{1}) = 1$. Hence $h(a) \neq 0$ for all $h \in \Delta$. On the other hand, if $a \notin \operatorname{Inv}(A)$, then

$$J := \{\, ba : b \in A \,\}$$

is a proper ideal in A. By Lemma 1.30, J is contained in a maximal ideal. Thus there is a $h \in \Delta$ such that $h(a) = 0$. This means that

$$a \in \operatorname{Inv}(A) \iff h(a) \neq 0 \text{ for all } h \in \Delta.$$

But then

$$0 \in \sigma(a) \iff h(a) = 0 \text{ for some } h \in \Delta.$$

Now we can obtain (d) by replacing a by $\lambda\mathbf{1} - a$ in the above. □

There are many algebras that arise in applications that do not have an identity. But the basic theory is much better suited to unital algebras and it is sometimes, but definitely not always, convenient to embed a nonunital algebra into a unital one.

Definition 1.31. Suppose that A is an algebra. Let $A^{\#}$ be the *vector space* direct sum $A \times \mathbf{C}$ equipped with the multiplication

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu).$$

It is quickly seen that $A^\#$ is a unital algebra. Furthermore, if A is a normed algebra, then we can make $A^\#$ into a normed algebra by

$$\|(a, \lambda)\| := \|a\| + |\lambda|. \tag{1.5}$$

Remark 1.32. While the norm given in (1.5) will suffice for the time being, it should be remarked that it is an arbitrary choice of sorts. Any submultiplicative norm extending the one on A would do. For example, we could have used $\|(a, \lambda)\| = \max\{\|a\|, |\lambda|\}$ instead. Later, when we get to C^*-algebras, we'll see that we need an even more subtle choice.

Remark 1.33. The notation $A^\#$ is nonstandard. The usual notation is A^1 or $\widetilde{A}$, but we are hunting C^*-algebras and want to reserve those notations for when we get to that territory.

Definition 1.34. If A does not have an identity, we define the *spectrum* to be the spectrum of a in $A^\#$:

$$\sigma(a) := \sigma_{A^\#}(a).$$

The spectral radius is defined as before:

$$\rho(a) = \sup\{\, |\lambda| : \lambda \in \sigma(a) \,\}.$$

It is important to notice that our definition of $\sigma(a)$ is purely algebraic and does not depend on which norm, if any, we choose to put on $A^\#$.

Notice that if A is not unital, then $0 \in \sigma_A(a)$ for all $a \in A$.

Remark 1.35. Notice that if $h \in \Delta(A)$, then there is a unique extension $\tilde{h}$ of h to an element of $\Delta(A^\#)$ given by

$$\tilde{h}(a, \lambda) = h(a) + \lambda.$$

Since $h \in \Delta(A)$ forces h to be nonzero, it follows that $\tilde{h}$ is nonzero as well. Since $\tilde{h}$ is automatically continuous by Theorem 1.29 on page 15, so is h.

Corollary 1.36. *Suppose that A is a nonunital commutative Banach algebra. If $a \in A$, then*

$$\sigma(a) = \{\, h(a) : h \in \Delta(A) \,\} \cup \{0\}.$$

Proof. Since the obvious map $h_\infty : A^\# \to \mathbf{C}$ is a complex homomorphism, we see that

$$\Delta(A^\#) = \{\, \tilde{h} : h \in \Delta(A) \,\} \cup \{h_\infty\}.$$

By Theorem 1.29(d),

$$\sigma(a) = \sigma_{A^\#}(a) = \{\, k(a) : k \in \Delta(A^\#) \,\},$$

and the conclusion follows. □

As it turns out (see E 1.3.4), not every commutative Banach algebra admits nonzero complex homomorphisms; that is, it is possible for $\Delta(A) = \emptyset$. This unfortunate circumstance makes it necessary to add the hypothesis that $\Delta(A) \neq \emptyset$ for general results. Thankfully, this will be unnecessary once we get to C^*-algebras.

Definition 1.37. Suppose that A is a commutative Banach algebra with $\Delta(A)$ nonempty. Then the *Gelfand transform* of $a \in A$ is the function

$$\hat{a} : \Delta(A) \to \mathbf{C}$$

given by $\hat{a}(h) := h(a)$. The Gelfand topology on $\Delta(A)$ is the smallest topology making each $\hat{a}$ continuous. The space $\Delta(A)$ is called *spectrum of* A. If A has an identity, then $\Delta(A)$ is also called the *maximal ideal space of* A.

It may be helpful to observe that we can view $\Delta(A)$ as a subset of the unit ball of the dual A^* of A. An important topology on A^* is the weak-$*$ topology which is the smallest topology making the functions $\varphi \mapsto \varphi(a)$ continuous for all $a \in A$. Recall that a net $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ converges to φ in the weak-$*$ topology on A^* if and only if $\varphi_\lambda(a) \to \varphi(a)$ for all $a \in A$.

Lemma 1.38. *The Gelfand topology on $\Delta(A)$ is the relative topology on $\Delta(A)$ viewed as a subset of A^* with the weak-$*$ topology.*

Proof. In the Gelfand topology, the sets

$$U(a, z, \epsilon) := \{\, h \in \Delta : |h(a) - z| < \epsilon \,\} \tag{1.6}$$

must be open for all $a \in A$, $z \in \mathbf{C}$ and $\epsilon > 0$. Since finite intersections of such sets form a basis for a topology, that topology must be the Gelfand topology. (Formally, the sets appearing in (1.6) form a *subbasis* for the Gelfand topology.)

Similarly weak star topology on A^* is generated by the sets

$$V(a, z, \epsilon) := \{\, \varphi \in A^* : |\varphi(a) - z| < \epsilon \,\}. \tag{1.7}$$

The result follows easily from the observation that $U(a, z, \epsilon) = V(a, z, \epsilon) \cap \Delta$. $\square$

Theorem 1.39. *Suppose that A is a commutative Banach algebra with nonempty spectrum Δ. Then Δ is a locally compact Hausdorff space and the Gelfand transform defines a norm-decreasing algebra homomorphism Φ of A into $C_0(\Delta)$. In fact, $\|\hat{a}\|_\infty = \rho(a)$. If A has a unit, then Δ is compact.*

Proof. We have $\Phi(a) = \hat{a}$ continuous on Δ by definition of the Gelfand topology, and Φ is easily seen to be an algebra homomorphism. By Theorem 1.29 and Corollary 1.36, either $\sigma(a) = \hat{a}(\Delta)$ or $\sigma(a) = \hat{a}(\Delta) \cup \{0\}$. In either case, $\|\hat{a}\|_\infty = \rho(a)$. If A is unital, then $\rho(a) \leq \|a\|$ by Corollary 1.26. Otherwise, we have $\rho(a) \leq \|a\|_{A^\#} = \|a\|_A$. Always, Φ is norm decreasing.

We still need to see that Δ is locally compact and that each $\hat{a}$ vanishes at infinity on Δ.

However the unit ball $K := \{\, \varphi \in A^* : \|\varphi\| \leq 1 \,\}$ is compact in the weak-$*$ topology (by the Banach-Alaoglu Theorem [Fol99, Theorem 5.18]), and Δ has the relative topology by Lemma 1.38. Hence the closure, Δ', of Δ in the weak-$*$ topology is compact and Hausdorff.

Suppose that $\varphi \in \Delta'$. Then there is a net $\{\, h_\lambda \,\}$ in Δ converging to φ in the weak-$*$ topology. Then $h_\lambda(a) \to \varphi(a)$ for all $a \in A$. Then

$$\varphi(ab) = \lim_\lambda h_\lambda(ab) = \lim_\lambda h_\lambda(a)h_\lambda(b) = \varphi(a)\varphi(b).$$

It follows that $\Delta' = \Delta \cup \{0\}$. Since the latter is compact, Δ is locally compact as well as Hausdorff.

If A is unital, then $h(\mathbf{1}) = 1$ for all $h \in \Delta$. Hence $\Delta' = \Delta$ and Δ is compact.

If $a \in A$ and $\epsilon > 0$, then

$$\{\, h \in \Delta : |\hat{a}(h)| \geq \epsilon \,\} = \{\, h \in \Delta : |h(a)| \geq \epsilon \,\} \tag{1.8}$$

must be closed in K. Hence the left-hand side of (1.8) is compact for all $\epsilon > 0$ and $\hat{a} \in C_0(\Delta)$ as required. □

If $A = C_0(X)$, with X locally compact, then our next result shows that we can identify $\Delta(A)$ with X and that the map Φ above is essentially the identity map. More formally, we have the following result.

Lemma 1.40. *Let X be a locally compact Hausdorff space, $A = C_0(X)$ and $\Delta = \Delta(C_0(X))$. Then every $h \in \Delta$ is a point evaluation; that is, $h = h_x$ where $h_x(f) := f(x)$ for some $x \in X$. Furthermore $x \mapsto h_x$ is a homeomorphism of X onto Δ. Hence the Gelfand transform induces an isometric isomorphism Φ of $C_0(X)$ onto $C_0(\Delta)$.*

Proof. In view of Theorem 1.39, it suffices to show that $x \mapsto h_x$ is a homeomorphism as claimed. Recall that $h_{x_i} \to h_x$ in Δ if and only if $h_{x_i}(f) \to h_x(f)$ for all $f \in C_0(X)$. But if $x_i \to x$ in X, then $h_{x_i}(f) = f(x_i) \to f(x) = h_x(f)$ since f is continuous. Hence $x \mapsto h_x$ is continuous.

On the other hand, if $h_{x_i} \to h_x$ and $x_i \not\to x$, then we can pass to a subnet, relabel, and assume that x has a neighborhood U disjoint from $\{x_i\}$. But there is a $f \in C_0(X)$ such that $f(x) = 1$ and such that f vanishes off U. Then $h_{x_i}(f) \not\to h_x(f)$ and we have a contradiction. Thus $x \mapsto h_x$ is a homeomorphism onto its range. We just have to see that it is surjective.

If X is compact, then the maximal ideals of $C(X)$ are all of the form $I_x = \{ f \in C(X) : f(x) = 0 \}$. Then it easily follows that all $h \in \Delta$ are point evaluations. If X is not compact and X^+ is its one-point compactification, then $C_0(X)$ can be identified with the maximal ideal of functions in $C(X^+)$ vanishing at the point ∞ (see E 1.1.9). Note that if $g \in C(X^+)$, then $g - g(\infty)1_{X^+} \in C_0(X)$. Hence if $h \in \Delta$, then we can define $\tilde{h} \in \Delta(C(X^+))$ by

$$\tilde{h}(g) = g(\infty) + h(g - g(\infty)1_{X^+}).$$

Hence h must be a point evaluation. This completes the proof.[5] $\square$

To Do: Exercises need work

Exercises

E 1.3.1. Show that $A^{\#}$ is a Banach algebra if A is (with the norm given by (1.5)).

E 1.3.2. Show that if A is a separable Banach space, then the unit ball of A^* is second countable in the weak-$*$ topology. Conclude that if A is a separable commutative Banach algebra with $\Delta(A)$ nonempty, then $\Delta(A)$ is a second countable locally compact Hausdorff space. (Hint: show that a countable family of the $V(a, z, \epsilon)$ from (1.7) suffice for a subbasis.)

E 1.3.3. Let A be the subset of 2×2-complex matrices of the form

$$A = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbf{C} \right\}. \tag{1.9}$$

[5]It is worth noting that $C(X^+)$ is not quite the same thing as $C_0(X)^{\#}$: although they are algebraically isomorphic, the norms are different. Instead, $C(X^+)$ is the algebra $C_0(X)^1$ from Lemma 2.15.

Show that A is a two dimensional unital commutative Banach algebra such that $\operatorname{Rad}(A) \neq \{0\}$.

E 1.3.4. Assume you remember enough measure theory to show that if $f, g \in L^1\bigl([0,1]\bigr)$, then

$$f * g(t) = \int_0^t f(t-s)g(s)\,ds \tag{1.10}$$

exists for almost all $t \in [0,1]$, and defines an element of $L^1\bigl([0,1]\bigr)$. Let A be the algebra consisting of the Banach space $L^1\bigl([0,1]\bigr)$ with multiplication defined by (1.10).

(a) Conclude that A is a commutative Banach algebra: that is, show that $f * g = g * f$, and that $\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1$.

(b) Let f_0 be the constant function $f_0(t) = 1$ for all $t \in [0,1]$. Show that

$$f_0^n(t) := f_0 * \cdots * f_0(t) = t^{n-1}/(n-1)!, \tag{1.11}$$

and hence,

$$\|f_0^n\|_1 = \frac{1}{n!}. \tag{1.12}$$

(c) Show that (1.11) implies that f_0 generates A as a Banach algebra: that is, $\operatorname{alg}(f)$ is norm dense. Conclude from (1.12) that the spectral radius $\rho(f)$ is zero for all $f \in A$.

(d) Conclude that A *has no nonzero complex homomorphisms.*

—Answer on page 134

E 1.3.5. Here we want to show that the disk algebra $A(D)$ (viewed as a subalgebra of $C(\mathbf{T})$) introduced in Example 1.17 is an example of a commutative Banach algebra A for which the Gelfand transform induces and injective isometric map of A onto a proper subalgebra of $C(\Delta)$. For convenience, let $p_n \in A$ be given by $p_n(z) = z^n$ for $n = 0, 1, 2, \ldots$, and let $\mathcal{P}$ be the subalgebra of polynomials spanned by the p_n.

(a) If $f \in A(D)$, let $\tilde{f}$ be its extension to $\overline{D}$. For each $z \in D$, define φ_z by $\varphi_z(f) = \tilde{f}(z)$. Observe that $\varphi_z \in \Delta$.

(b) Let $\Psi : \overline{D} \to \Delta$ be given by $\Psi(z) = \varphi_z$. Observe that Ψ is injective. (Consider p_1.)

(c) If $f \in A$ and $0 < r < 1$, then let $f_r(z) := f(rz)$. Show that $f_r \to f$ in A as $r \to 1$.

(d) Conclude that $\mathcal{P}$ is dense in A. (Hint: show that $f_r \in \overline{\mathcal{P}}$ for all $0 < r < 1$.)

(e) Now show that Ψ is surjective. (Hint: suppose that $h \in \Delta$. Then show that $h = \varphi_z$ where $z = h(p_1)$.)

(f) Show that Ψ is a homeomorphism. (Hint: Ψ is clearly continuous and both $\overline{D}$ and Δ are compact and Hausdorff.)

(g) Observe that if we use the above to identify Δ and $\overline{D}$, then the Gelfand transform maps $A(D)$ isometrically onto a proper subalgebra of $C(\overline{D})$. (Namely the algebra B from Example 1.17.)

E 1.3.6. In this problem, we want to prove an old result to due Wiener about functions with absolutely converent Fourier series using the machinery of Gelfand theory. Recall that if $\varphi \in C(\mathbf{T})$, then the *Fourier coefficients* of φ are given by[6]

$$\check{\varphi}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(e^{it}) e^{-int} \, dt.$$

In some cases — for example if φ has two continuous derivatives — the Fourier coefficients are *absolutely convergent* in the sense that $n \mapsto \check{\varphi}(n)$ defines an element of $\ell^1(\mathbf{Z})$.

We aim to prove the following:

Theorem: (Wiener) *Suppose the $\varphi \in C(\mathbf{T})$ never vanishes and has an absolutely convergent Fourier series. Then $\psi := 1/\varphi$ also has an absolutely convergent Fourier series.*

I suggest the following strategy.

(a) Show that $\ell^1(\mathbf{Z})$ is a unital commutative Banach algebra under convolution as asserted in Example 1.16. (Here, $1_{\ell^1(\mathbf{Z})} = \mathbb{1}_{\{0\}}$.[7])

[6] I've used $\check{\varphi}$ in place of the traditional $\hat{\varphi}$ to distinguish it from the (other) Gelfand transform to be used in the problem.

[7] If S is a subset of X, I use $\mathbb{1}_S$ for the characteristic function of S, which takes the value 1 on S, and 0 otherwise.

(b) Let $\Delta = \Delta(\ell^1(\mathbf{Z}))$ be the maximal ideal space of $\ell^1(\mathbf{Z})$ equipped with its compact, Hausdorff Gelfand topology. If $z \in \mathbf{T}$, then define $h_z : A \to \mathbf{C}$ by

$$h_z(f) = \sum_{n=-\infty}^{\infty} f(n)z^n.$$

Show that $h_z \in \Delta$.

(c) Let $w = \mathbb{1}_1 \in \ell^1(\mathbf{Z})$. If $h \in \Delta$, then show that $h = h_z$ where $z = h(w)$. (Hint: If $f \in \ell^1(\mathbf{Z})$, then $f = \sum_{n=-\infty}^{\infty} f(n)w^n$ in norm in $\ell^1(\mathbf{Z})$, where for example, $w^2 = w * w = \mathbb{1}_2$ and $w^{-1} = \mathbb{1}_{-1}$.)

(d) Show that $z \mapsto h_z$ is a homeomorphism Φ of $\mathbf{T}$ onto Δ. (Hint: Since both $\mathbf{T}$ and Δ are compact Hausdorff sets, it suffices to see that Φ is a continuous bijection. To show that Φ is continuous, observe that functions of the form $\sum_{n=-N}^{n=N} f(n)w^n$ are dense in $\ell^1(\mathbf{Z})$.)

(e) Since we can identify $\mathbf{T}$ with Δ, if $f \in \ell^1(\mathbf{Z})$, we will view the Gelfand transform of f as a continuous function on $\mathbf{T}$. (So that we write $\hat{f}(z)$ in place of $\hat{f}(h_z)$.) Show that if $\varphi = \hat{f}$ for some $f \in \ell^1(\mathbf{Z})$, then $\check{\varphi} = f$.

(f) Conclude that the image $\mathfrak{A}$ of $\ell^1(\mathbf{Z})$ in $C(\mathbf{T})$ under the Gelfand transform is exactly the set of φ in $C(\mathbf{T})$ whose Fourier coefficients are absolutely convergent. (That is, $\mathfrak{A}$ is the collection of $\varphi \in C(\mathbf{T})$ such that $n \mapsto \check{\varphi}(n)$ is in $\ell^1(\mathbf{Z})$.)

(g) Now prove Wiener's Theorem as stated above. (Hint: More or less by assumption, $\varphi = \hat{f}$ for some f in $\ell^1(\mathbf{Z})$. Show that f must be invertible in $\ell^1(\mathbf{Z})$ and consider the Gelfand transform of the inverse of f.)

—Answer on page 135

1.4 Examples: Abelian Harmonic Analysis

This section is meant purely to illustrate some of the power and interest of the Gelfand theory. It is a beautiful subject, but it is not the direction we are headed so I won't be giving many proofs. Some of the details can be found in [Wil07, §1.4]. The authoritative source is the venerable [Loo53]. The basic results can be stated for any locally compact abelian group G. Such a group always has a Borel measure λ — called a Haar measure — that is translation invariant in that $\lambda(E + g) = \lambda(E)$

for all Borel sets E and which is nontrivial in that $\lambda(V) > 0$ for all open sets V. The translation invariance is equivalent to the integral equation

$$\int_G f(s+r)\,d\lambda(s) = \int_G f(s)\,d\lambda(s)$$

for all integrable f and elements $r \in G$. Then we want to make the Banach space $A = L^1(G, \lambda)$ into a Banach algebra. At this point, it would be fine to concentrate on just three examples.

Example 1.41 (The Real Line). Let $G = \mathbf{R}$ and λ Lebesgue measure. Then

$$\int_{\mathbf{R}} f(s)\,d\lambda(s) = \int_{-\infty}^{\infty} f(s)\,ds.$$

Example 1.42 (The Integers). Let $G = \mathbf{Z}$. Then λ is just counting measure. Then

$$\int_{\mathbf{Z}} f(s)\,d\lambda(s) = \sum_{n\in\mathbf{Z}} f(n) = \sum_{n=-\infty}^{\infty} f(n).$$

Example 1.43. Let $G = \mathbf{T}$. Then we can take normalized Lebesgue measure for λ so that

$$\int_{\mathbf{T}} f(s)\,d\lambda(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\,dt.$$

In any event, we can make $A = L^1(G, \lambda)$ into a Banach algebra by defining multiplication via *convolution*:

$$f * g(s) := \int_G f(r)g(s-r)\,d\lambda(s). \tag{1.13}$$

I am purposely playing fast and loose with the niceties of measure theory. For example, if f and g are integrable functions, then one has to argue that (1.13) is a well-defined function at least almost everywhere. Then one has to justify using Fubini's Theorem in the next calculation. We'll just pretend we all remember enough measure theory to justify the manipulations used here. Then

$$\begin{aligned}\|f * g\|_1 &= \int_G |f * g(s)|\,d\lambda(s)\\ &\le \int_G \int_G |f(r)g(s-r)\,d\lambda(r)\,d\lambda(s)\\ &= \int_G |f(r)| \int_G |g(s-r)|\,d\lambda(s)\,d\lambda(r)\\ &= \|f\|_1\|g\|_1.\end{aligned}$$

The point is that with respect to convolution, $L^1(G, \lambda)$ is a commutative Banach algebra. As it turns out, $L^1(G, \lambda)$ has an identity only when G is discrete in which case the identity is the indicator function, $\mathbb{1}_{\{e\}}$, of the group identity e.

A continuous group homomorphism $\gamma : G \to \mathbf{T}$ is called a *character* of G. The collection of characters, $\hat{G}$, is a group, and it is not hard to check that each $\gamma \in \hat{G}$ defines a complex homomorphism $h_\gamma \in \Delta(L^1(G))$ via

$$h_\gamma(f) := \int_G f(s)\gamma(s)\, d\lambda(s).$$

In fact, every $h \in \Delta(L^1(G))$ is of this form. (A special case of this was worked out in E 1.3.6. For the general case, see [Wil07, Proposition 1.76].) Therefore, we can identify $\Delta(L^1(G))$ with $\hat{G}$. Furthermore, under this identification, the Gelfand topology on $\hat{G}$ is just the compact-open topology: $\gamma_i \to \gamma$ if and only if $\{\gamma_i\}$ converges uniformly to γ on each compact subset of G. Note that this makes $\hat{G}$ into a locally compact abelian group. In this case, the Gelfand transform is called the *Fourier transform* and we will write

$$\hat{f}(\gamma) \quad \text{in place of} \quad \hat{f}(h_\gamma) = h_\gamma(f).$$

To get back to reality, let's look at the Fourier transform in our three basic examples: 1.41, 1.42 and 1.43.

If $G = \mathbf{R}$, then it is not so hard to check that every character $\gamma \in \hat{R}$ is of the form $\gamma_y(x) = e^{-ixy}$ for some $y \in \mathbf{R}$. Then we can identify $\hat{R}$ with $\mathbf{R}$ as a set and it is not so difficult to see that the identification is topological as well. Then the Fourier transform is the classical one:

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy}\, dx.$$

Since the Fourier transform is just the Gelfand transform (modulo our identifications), it follows from the Gelfand theory (Theorem 1.39) that $\hat{f} \in C_0(\mathbf{R})$ and $\|\hat{f}\|_\infty \le \|f\|_1$.

If $G = \mathbf{Z}$, then each $\gamma \in \hat{\mathbf{Z}}$ is determined by its value at $1 \in \mathbf{Z}$. Thus $\gamma(n) = z^n$ for some $z \in \mathbf{T}$ and we can identify $\hat{\mathbf{Z}}$ with $\mathbf{T}$ (even topologically). If $f \in \ell^1(\mathbf{Z})$, then

$$\hat{f}(z) = \sum_{n=-\infty}^{\infty} f(n)z^n.$$

Here $\hat{f} \in C(\mathbf{T})$ and $\|\hat{f}\|_\infty \le \|f\|_1$.

If $G = \mathbf{T}$, then each $\gamma \in \hat{\mathbf{T}}$ turns out to be of the form $\gamma_n(z) = z^{-n}$. Therefore, $\hat{\mathbf{T}}$ can be identified with $\mathbf{Z}$ and we have

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int} \, dt.$$

In this case, $\hat{f} \in c_0(\mathbf{Z})$ and each $|\hat{f}(n)|$ is bounded by $\|f\|_1$.

Remark 1.44. It is no accident that $\hat{\hat{\mathbf{Z}}}$ can be identified with $\mathbf{Z}$. The Pontrjagin Duality Theorem (see for example, [Fol95, Theorem 4.31]) implies that the character group of $\hat{G}$ can be identified with G in the obvious way.

Remark 1.45. It is worth noting that in each of our three basic examples, the Fourier transform — really the Gelfand transform — gives injective algebra homomorphisms of $A = L^1(G)$ onto dense subalgebras of $C_0(\hat{G})$.

To Do: Discuss closed ideals in $L^1(G, \lambda)$. It is easy to produce ideals corresponding to closed subsets of the dual $\hat{G}$ [Fol95, Theorem 4.51]. If G is compact, then these are the only closed ideals in $L^1(G, \lambda)$ [Fol95, Theorem 4.53]. But if G is not compact, then there are always other sorts of closed ideals by results due to Malliavin. See [Rud62, §7.6] for a proof and further references. Apparently, a straightforward example is known for $G = \mathbf{R}^3$ or even $G = \mathbf{R}^n$ with $n \geq 3$: see [Fol95, Theorem 4.54] or [Rud62, §7.3].

Basically [Fol95, §4.5] and [Rud62, Chap. 7] are devoted to ideals in $L^1(G, \lambda)$.

Chapter 2

Getting it Right: the C^*-Norm Identity

In the previous chapter, we developed the Gelfand theory and played a bit with commutative Banach algebras. In particular, if A is a unital commutative Banach algebra with maximal ideal space Δ, then the Gelfand transform gives us an algebra homormorphism $\Phi : A \to C(\Delta)$. The following examples give us an idea of just how effective a tool the Gelfand transform is for "understanding" A.

Example 2.1. Let A_0 be the algebra from E 1.3.4 on page 21, and let $A = {A_0}^\#$. In this case, $\Delta = \{h_\infty\}$ and Φ has kernel A_0. This is about as bad as it gets.

Example 2.2. Let $A = \ell^1(\mathbf{Z})$. Then as we saw in E 1.3.6 on page 22, we can identify Δ with $\mathbf{T}$. In this case Φ is injective and has dense range consisting of those functions in $C(\mathbf{T})$ that have absolutely convergent Fourier series. Note that Φ is not isometric.

Example 2.3. Let $A = A(\mathbf{D})$ be the disk algebra (see Example 1.17). Then we can identify Δ with the closed unit disk $\overline{\mathbf{D}}$ and Φ is essentially the identity map. Therefore Φ maps A isometrically onto a proper (closed) subalgebra of $C(\overline{\mathbf{D}})$.

Example 2.4. [Getting it Right] Let X be a compact Hausdorff space and $A = C(X)$. Then as we saw in Lemma 1.40, we can identify Δ with X, and Φ is an isometric isomorphism.

Examples 2.1, 2.2 and 2.3 all have defects. But Example 2.4 is as good as Example 2.1 is pathological. We need to see for what sort of commutative Banach algebras the Gelfand transform induces an isometric isomorphism.

2.1 Banach $*$-algebras

Recall that if $T \in B(\mathcal{H})$, then T has an *adjoint* $T^* \in B(\mathcal{H})$ characterized by

$$(Th \mid k) = (h \mid T^*k) \quad \text{for all } h, k \in \mathcal{H}.$$

For example, after identifying M_n with $B(\mathbf{C}^n)$ via any orthonormal basis, the adjoint of T is given by its conjugate transpose.

We know from elementary linear algebra that the adjoint plays a significant role in understanding linear maps. For example, only normal operators are orthogonally diagonalizable. So it may not be so surprising that we will need an analogue of an adjoint to find well behaved algebras.

Definition 2.5. An *involution* on an algebra A is a map $a \mapsto a^*$ of A onto itself such that

(a) $(a^*)^* = a$,
(b) $(ab)^* = b^*a^*$ and
(c) $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$

for all $a, b \in A$ and $\lambda \in \mathbf{C}$.

Remark 2.6. Note that if $\mathbf{1} \in A$, then property (b) implies that we must have $\mathbf{1}^* = \mathbf{1}$.

Of course the proto-typical example of an involution on a Banach algebra is the adjoint operation on $B(\mathcal{H})$. Another important example is complex conjugation in $C_0(X)$: $f^*(x) := \overline{f(x)}$.

Definition 2.7. A $*$-algebra is an algebra A together with an involution. A *Banach $*$-algebra* is a Banach algebra A together with an isometric involution. (That is, we insist that $\|a^*\| = \|a\|$ for all $a \in A$.) An algebra homomorphism $\varphi : A \to B$ between $*$-algebras is called a $*$-homomorphism if $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.

Remark 2.8. Not everyone insists that the $*$-operation should be isometric. For example, it is certainly interesting to just consider algebras where it is continuous. But our model is the adjoint operation in $B(\mathcal{H})$.

Definition 2.9. A Banach $*$-algebra is called a *C^*-algebra* if $\|a^*a\| = \|a\|^2$ for all $a \in A$.

Example 2.10. If X is a locally compact Hausdorff space, then $A = C_0(X)$ is a commutative C^*-algebra when equipped with the involution given by complex conjugation.

As we shall see, every commutative C^*-algebra arises as in Example 2.10. For the canonical noncommutative example, we need the following.

Lemma 2.11. *If $\mathcal{H}$ is a Hilbert space and $T \in B(\mathcal{H})$, then $\|T^*\| = \|T\|$ and $\|T^*T\| = \|T\|^2$.*

Proof. The operator norm is submultiplicative. Hence $\|T^*T\| \leq \|T^*\|\|T\|$. If $\|h\| = 1$, then the Cauchy-Schwarz inequality implies that $(T^*Th \mid h) \leq \|T^*T\|$. Therefore

$$\|T^*\|\|T\| \geq \|T^*T\| \geq \sup_{\|h\|=1} (T^*Th \mid h) = \sup_{\|h\|=1} \|Th\|^2 = \|T\|^2.$$

Hence $\|T^*\| \geq \|T\|$. Replacing T by T^* gives $\|T^*\| = \|T\|$. Therefore $\|T\|^2 \geq \|T^*T\| \geq \|T\|^2$ and we're done. □

Example 2.12. Any norm closed $*$-subalgebra of $B(\mathcal{H})$ is a C^*-algebra.

Example 2.12 is generic in the sense that every C^*-algebra is isometrically $*$-isomorphic to a closed $*$-subalgebra of some $B(\mathcal{H})$.[1]

Our next result is the promised characterization of commutative C^*-algebras. It is called the *Abstract Spectral Theorem* since it leads naturally to the functional calculus.[2]

Theorem 2.13 (Abstract Spectral Theorem)**.** *Suppose that A is a commutative C^*-algebra.[3] Then the spectrum, Δ, of A is nonempty and the Gelfand transform defines an isometric $*$-isomorphism, Φ, of A onto $C_0(\Delta)$.*

Remark 2.14. Saying that Φ is a $*$-isomorphism is the assertion that $\widehat{a^*}(h) = \overline{\hat{a}(h)}$ for all $h \in \Delta$. Since the Gelfand transform is defined for commutative Banach algebras, it is a pleasant bit of serendipity that it automatically respects the $*$-structure of a commutative C^*-algebra.

Before embarking on the proof of Theorem 2.13, a couple of preliminary results will be helpful. First recall that is A is any algebra (unital or not), then we can make the vector space direct sum $A \times \mathbf{C}$ into a unital algebra (see Definition 1.31). Clearly, if A is a $*$-algebra, then $A^\#$ becomes a $*$-algebra if we define $(a, \lambda)^* = (a^*, \bar{\lambda})$.

[1] Do I ever prove this? Of course, we could cite [RW98, Theorem A.11].

[2] References is spectral theorem for compact operators and ???

[3] I have been told that there have been debates over whether the trivial vector space $\{0\}$ should be considered as a C^*-algebra. I suppose the answer is "sometimes" as it might be convenient to talk about C^*-bundles, whatever they are, were some of the fibres are $\{0\}$. But I simply can't bring myself to write "nonzero C^*-algebra" in the statement of results. So I hope it is clear that that in cases like this, that A is not the zero C^*-algebra — if there is such a thing.

Lemma 2.15. *If A is a C^*-algebra, then the $*$-algebra $A^\#$ admits a norm making it into a unital C^*-algebra, denoted A^1, such that $a \mapsto (a,0)$ is isometric.*

Proof. This is a homework exercise (E 2.1.4). □

Proof of Theorem 2.13. To begin with, assume that A has an identity. Then Δ is nonempty by Theorem 1.29 on page 15 and Φ is an algebra homomorphism of A into $C(\Delta)$. In view of Remark 2.14, to see that Φ is a $*$-homomorphism, we just need to see that

$$h(a^*) = \overline{h(a)} \quad \text{for all } h \in \Delta. \tag{2.1}$$

However, if $a \in A$, then

$$a = \frac{a+a^*}{2} + \frac{a-a^*}{2} = \frac{a+a^*}{2} + i\frac{(-ia)+(-ia)^*}{2} = x_1 + ix_2 \tag{2.2}$$

where each x_j is *self-adjoint*.[4] Hence, in order to establish (2.1) it will suffice to see that $h(a)$ is real whenever a is self-adjoint.

To this end, let $a = a^*$ and fix $h \in \Delta$. For each $t \in \mathbf{R}$, define

$$u_t = \exp(ita) := \sum_{n=0}^{\infty} \frac{(ita)^n}{n!}.$$

Note that since $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent in any Banach algebra, $\exp(x)$ makes perfectly good sense. Moreover, some manipulation with power series should convince you that if $xy = yx$, then $\exp(x+y) = \exp(x)\exp(y)$. Furthermore, since the involution is isometric in a Banach $*$-algebra, we have $\exp(x)^* = \exp(x^*)$ in any Banach $*$-algebra. Thus,

$$\|u_t\|^2 = \|u_t^* u_t\| = \|\exp(-ita)\exp(ita)\| = \|\exp(0)\| = 1.$$

However, since h is bounded, $h(u_t) = \exp(ith(a))$, and since $\|u_t\| \le 1$, we must have $|h(u_t)| \le 1$. Therefore for all $t \in \mathbf{R}$, we have

$$\exp\bigl(t\operatorname{Re}\bigl(ih(a)\bigr)\bigr) = \bigl|\exp(ith(a))\bigr| = |h(u_t)| \le 1.$$

This can only happen if $\operatorname{Re}(ih(a)) = 0$. Thus Φ must be a $*$-homomorphism as claimed.

[4] An element x in a $*$-algebra A is self-adjoint if $x^* = x$. The collection of self-adjoint elements in A is denoted by $A_{s.a.}$.

To see that Φ is isometric, note that if $a = a^*$, then $\|a^2\| = \|a\|^2$ by the C^*-norm identity. By induction, $\|a^{2^n}\| = \|a\|^{2^n}$ for any $n \geq 1$. But Theorem 1.24 and 1.39 imply that

$$\|\hat{a}\|_\infty = \rho(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|.$$

Thus, in general,

$$\|\hat{a}\|_\infty^2 = \|\bar{\hat{a}}\hat{a}\| = \|a^*a\| = \|a\|^2.$$

Thus Φ is isometric.

It follows that Φ maps A isometrically onto a subalgebra $\mathfrak{A}$ of $C(\Delta)$ that separates points, is closed under complex conjugation and contains the constant functions. Thus $\mathfrak{A}$ is dense by the Stone-Weierstrass Theorem. Since Φ is isometric, $\mathfrak{A}$ is complete and therefore closed in $C(\Delta)$. Hence $\mathfrak{A} = C(\Delta)$, and Φ is surjective.

If A is not unital, we can apply the above to A^1 and obtain an isometric $*$-isomorphism Φ^1 of A^1 onto $C(\Delta(A^1))$. Recall that $\Delta(A^1) = \{\, \tilde{h} : h \in \Delta(A)\,\} \cup \{h_\infty\}$, where $h_\infty : A^1 \to \mathbf{C}$ is given by $h_\infty((a,\lambda)) = \lambda$ and $\tilde{h}$ is the unique extension of $h \in \Delta(A)$ to an element of $\Delta(A^1)$.

Suppose that $b \in A^1$. Then $b \in A$ if and only if $\hat{b}(h_\infty) = 0$. Since Φ^1 is isometric and $A \neq \{0\}$, we must have $\Delta(A) \neq \emptyset$. Also observe that if $\Phi : A \to C_0(\Delta(A))$ is the homomorphism induced by the Gelfand transform on A, then

$$\Phi(a)(h) = \Phi^1(a)(\tilde{h}).$$

In particular, follows that Φ is an isometric $*$-homomorphism. Therefore the image of Φ is a complete, hence closed self-adjoint subalgebra of $C_0(\Delta(A))$ that separates points and has no common zeros. Hence Φ is surjective. This completes the proof. $\square$

Exercises

E 2.1.1. Suppose that A is a C^*-algebra.

(a) Suppose that $e \in A$ satisfies $xe = x$ for all $x \in A$. Show that $e = e^*$ and that $\|e\| = 1$. Conclude that e is a unit for A.

(b) Show that for any $x \in A$, $\|x\| = \sup_{\|y\|\leq 1} \|xy\|$. (Do *not* assume that A has an approximate identity.)

—Answer on page 135

E 2.1.2. Suppose that A is a Banach algebra with an involution $x \mapsto x^*$ that satisfies $\|x\|^2 \leq \|x^*x\|$. Then show that A is a Banach $*$-algebra (i.e., $\|x^*\| = \|x\|$). In fact, show that A is a C^*-algebra.

—Answer on page 135

E 2.1.3. Let I be a set and suppose that for each $i \in I$, A_i is a C^*-algebra. Let $\bigoplus_{i\in I} A_i$ be the subset of the direct product $\prod_{i\in I} A_i$ consisting of those $a \in \prod_{i\in I} A_i$ such that $\|a\| := \sup_{i\in I} \|a_i\| < \infty$. Show that $\bigl(\bigoplus_{i\in I} A_i, \|\cdot\|\bigr)$ is a C^*-algebra with respect to the usual pointwise operations:

$$\begin{aligned}(a+\lambda b)(i) &:= a(i) + \lambda b(i)\\ (ab)(i) &:= a(i)b(i)\\ a^*(i) &:= a(i)^*.\end{aligned}$$

We call $\bigoplus_{i\in I} A_i$ the *direct sum* of the $\{\, A_i \,\}_{i\in I}$.

—Answer on page 135

E 2.1.4. Let A^1 be the vector space direct sum $A \oplus \mathbf{C}$ with the $*$-algebra structure given by

$$\begin{aligned}(a,\lambda)(b,\mu) &:= (ab + \lambda b + \mu a, \lambda\mu)\\ (a,\lambda)^* &:= (a^*, \bar{\lambda}).\end{aligned}$$

Show that there is a norm on A^1 making it into a C^*-algebra such that the natural embedding of A into A^1 is isometric. (Hint: If $1 \in A$, then show that $(a,\lambda) \mapsto (a + \lambda 1_A, \lambda)$ is a $*$-isomorphism of A^1 onto the C^*-algebra direct sum of A and $\mathbf{C}$. If $1 \notin A$, then for each $a \in A$, let L_a be the linear operator on A defined by left-multiplication by a: $L_a(x) = ax$. Then show that the collection B of operators on A of the form $\lambda I + L_a$ is a C^*-algebra with respect to the operator norm, and that $a \mapsto L_a$ is an isometric $*$-isomorphism.)

—Answer on page 136

2.2 The Functional Calculus

In general Banach algebras, then notion of the spectrum of an element is touchy — in can depend on what subalgebra we wish to view the element as belonging to. That is, if B is a subalgebra of A and if $\mathbf{1}_A \in B$, then for $b \in B$ it can happen that $\sigma_A(b)$ is a proper subset of $\sigma_B(b)$. (This is because it is easier to be invertible in A than in B.) For a specific example, see E 2.2.5. (Nevertheless, there are restrictions on how different these sets can be — see [Rud91, Theorem 10.18].) Fortunately, in a C^*-algebra, there is no ambiguity. (This will be important for the functional calculus to be developed shortly.)

Theorem 2.16 (Spectral Permanence)**.** *Suppose that A is a unital C^*-algebra and that B is a C^*-subalgebra containing the identity of A. Then for all $b \in B$, we have*

$$\sigma_B(b) = \sigma_A(b).$$

Proof. We clearly have $\sigma_A(b) \subset \sigma_B(b)$ and a bit of reflection shows that it will suffice to prove that whenever $b \in B$ in invertible in A, then it is already invertible in B.

In fact, I claim it suffices to show this for self-adjoint elements. To see this, suppose $b \in \operatorname{Inv}(A) \cap B$. Then $b^* \in \operatorname{Inv}(A)$ as is b^*b. Then

$$\big((b^*b)^{-1}b^*\big)b = \mathbf{1}_A,$$

and $b^{-1} = (b^*b)^{-1}b^*$. Hence if $(b^*b)^{-1} \in B$, then so is b^{-1}. This establishes the claim.

So, assume $b = b^*$ and that $b \in \operatorname{Inv}(A)$. Let

$$C = C^*(\{b, b^{-1}\}) \subset A.$$

Also, let

$$D = C^*(\{1, b\}) \subset C \cap B.$$

Since $(b^{-1})^* = b^{-1}$, C is the closure of the algebra generated by $\{b, b^{-1}\}$. In particular, C is commutative and unital. Hence

$$C = C(\Delta)$$

and $C(\Delta)$ is generated by the functions $\hat{b}$ and $\hat{c} = 1/\hat{b}$. But $\hat{b}(h) = \hat{b}(h')$ implies that $\hat{c}(h) = \hat{c}(h')$. Therefore $\hat{b}$ must separate points of Δ. Since $\hat{b}$ never vanishes, it must generate $C(\Delta)$ by Stone-Weierstrass. Thus the image of D in $C(\Delta)$ must be all of $C(\Delta)$. Hence $D = C$ and $b^{-1} \in D \subset B$ as required. □

Remark 2.17. If S is a sub*set* of a C^*-algebra A, then the intersection, $C^*(S)$, of all C^*-subalgebras of A containing S is itself a C^*-subalgebra of A containing S called *the C^*-subalgebra of A generated by S.*

We say that an element a in a C^*-algebra is *normal* if $a^*a = aa^*$.

Theorem 2.18 (The Functional Calculus)**.** *Suppose that A is a unital C^*-algebra and that a is a normal element in A. Then there is an isometric $*$-isomorphism*

$$\Psi : C\big(\sigma(a)\big) \to C^*\big(\{\mathbf{1}_A, a\}\big) \subset A$$

such that $\Psi(\mathrm{id}) = a$, where $\mathrm{id}(\lambda) = \lambda$ for all $\lambda \in \sigma(a)$.

Proof. Let $B := C^*\big(\{\mathbf{1}_A, a\}\big)$. Then B is a unital commutative C^*-algebra (see E 2.2.1) and the Gelfand transform gives us an isometric $*$-isomorphism

$$\Phi : B \to C(\Delta).$$

Furthermore, since two complex homomorphisms that agree on a must be equal on B, $\tau : \Delta \to \sigma(a)$ given by $\tau(h) := h(a)$ is a continuous bijection. Since Δ and $\sigma(a)$ are compact and Hausdorff, τ is a homeomorphism. Then we get an isometric $*$-isomorphism

$$\Theta : C\big(\sigma(a)\big) \to C(\Delta)$$

by $\Theta(f)(h) = f\big(\tau(h)\big) = f\big(h(a)\big) = f\big(\hat{a}(h)\big)$.

Let $\Psi = \Phi^{-1} \circ \Theta$. Since $\Theta(\mathrm{id}) = \hat{a}$, $\Phi^{-1}(\Theta(\mathrm{id}) = \Phi^{-1}(\hat{a}) = a$ as required. □

We are going to see soon that the "functional calculus" of Theorem 2.18 is very powerful. To begin to see why, we need some simple observations and some useful notation.

First, suppose that p is a polynomial in $\mathbf{C}[z]$; say

$$p(z) = \lambda_0 + \lambda_1 z + \cdots + \lambda_n z^n.$$

Then everyone agrees that if $a \in A$, we get an element

$$p(a) = \lambda_0 \mathbf{1}_A + \lambda_1 a + \cdots + \lambda_n a^n$$

which belongs to A provided $\mathbf{1}_A \in A$. Since $\Psi : C\big(\sigma(a)\big) \to C^*\big(\{\mathbf{1}_A, a\}\big)$ is an algebra homomorphism such that $\Psi(\mathrm{id}) = a$, we certainly have

$$\Psi(p) = p(a).$$

More generally, if $q \in C[z, w]$ is a polynomial in two variables and if we define $g \in C\big(\sigma(a)\big)$ by $g(\lambda) = q(\lambda, \bar{\lambda})$, then since Ψ is a $*$-homomorphism we have

$$\Psi(g) = q(a, a^*).$$

For this reason, if $f \in C\big(\sigma(a)\big)$, then we usually write $f(a)$ in place of $\Psi(f)$.

One of the purposes of the functional calculus is produce elements in a C^*-algebra possessing nice properties using just the algebra in a function algebra. We also want to be able to do this when the C^*-algebra in question does not have an identity. For that, we have the following convenient corollary. Note that if $0 \in \sigma(a)$, then we can identify $C_0\big(\sigma(a) \setminus \{0\}\big)$ with the continuous functions on $\sigma(a)$ which vanish at 0—consider the case where 0 is isolated in $\sigma(a)$ separately. Of course, if $0 \notin \sigma(a)$, then $C_0\big(\sigma(a) \setminus \{0\}\big) = C\big(\sigma(a)\big)$.

Corollary 2.19. *Suppose that A is a C^*-algebra and that $a \in A$ is normal. Then there is an isometric $*$-isomorphism of $C_0\big(\sigma(a) \setminus \{0\}\big)$ onto $C^*\big(\{a\}\big) \subset A$ taking the identity function to a.*

Proof. This a homework exercise. (Hint: It suffices to consider the case where $0 \in \sigma(a)$. Let $\widetilde{A}$ be A if A has an identity and A^1 otherwise. Consider the map Ψ mapping $C\big(\sigma(a)\big)$ into $\widetilde{A}$, and notice that $C_0\big(\sigma(a) \setminus \{0\}\big)$ is generated by id.) □

We know that if $T \in B(\mathcal{H})$ and $\dim \mathcal{H} = \infty$, then elements of $\sigma(T)$ need not be eigenvalues. But it T is normal, then this is "almost true".

Corollary 2.20 (Approximate Eigenvectors)**.** *Suppose that $T \in B(\mathcal{H})$ is normal and that $\lambda \in \sigma(T)$. Then there are unit vectors $\{h_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $(T - \lambda I)h_n \to 0$ in $\mathcal{H}$.*

Proof. Replacing T by $T - \lambda I$, it suffices to consider $\lambda = 0$. Thus, we assume $0 \in \sigma(T)$ and let $\Psi : C\big(\sigma(T)\big) \to B(\mathcal{H})$ be the functional calculus map (so that $\Psi(\mathrm{id}) = T$). Choose $f_n \in C\big(\sigma(T)\big)$ such that $f_n(0) = 1$, $\|f\|_\infty = 1$ and such that f_n vanishes off $B_{\frac{1}{n}}(0) = \{\, z \in \mathbf{C} : |z| < \frac{1}{n} \,\}$. Let $f = \mathrm{id}$. The point is that $ff_n \to 0$ in $C\big(\sigma(T)\big)$. Since Ψ is an isometric isomorphism,

$$\Psi(ff_n) = \Psi(f)\Psi(f_n) = Tf_n(T) \to 0 \quad \text{in } B(\mathcal{H}).$$

Since $\|f_n(T)\| = 1$, there is a $k_n \in \mathcal{H}$ such that $h_n := f_n(T)k_n$ is a unit vector and $\|k_n\| \le 2$. Then

$$Th_n = \Psi(ff_n)k_n \to 0$$

as required. □

Remark 2.21 (Polarization). Since we are always working over the complex field, we often can take advantage of the *polarization identity*:

$$(Th \mid k) = \frac{1}{4}\sum_{n=0}^{3} i^n \big(T(h+i^n k) \mid h+i^n k\big)$$

where, of course, $i=\sqrt{-1}$.

Note that polarization has the immediate consequence that if a linear operator T on a complex Hilbert space satisfies $(Th \mid h)=0$ for all $h\in\mathcal{H}$, then $T=0$. This is false over the reals: consider the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as an operator on $\mathbf{R}^2$.

Lemma 2.22. *If $T\in B(\mathcal{H})$ is such that $(Th \mid h)$ is real for all $h\in\mathcal{H}$, then T is self-adjoint.*

Proof. Since $(Th \mid h)$ is real, $(Th \mid h)=(h \mid Th)=(T^*h \mid h)$. But by the comments above $T-T^*$ is the zero operator. □

An operator T in $B(\mathcal{H})$ is called a *positive operator* if $(Th \mid h)\geq 0$ for all $h\in\mathcal{H}$. Positive operators play a central role in operator theory and we will set about finding the corresponding notion in a general C^*-algebra in the next section.

Theorem 2.23. *Suppose that $T\in B(\mathcal{H})$. Then the following are equivalent.*

(a) *T is a positive operator.*
(b) *T is normal and $\sigma(T)\subset[0,\infty)$.*
(c) *$T=R^*R$ for some $R\in B(\mathcal{H})$.*

Furthermore, if T is positive, then T has a unique positive square root $T^{\frac12}$ and $T^{\frac12}$ can be approximated in norm by polynomials in T.

Proof. Suppose that T is positive and $\lambda\in\sigma(T)$. Then T is normal, and in fact self-adjoint, by Lemma 2.22. By Corollary 2.20 on the previous page we can find a sequence of unit vectors $\{h_n\}$ (a.k.a. approximate eigenvectors) such that $(T-\lambda I)h_n\to 0$. Thus

$$(Th_n \mid h_n)\to\lambda,$$

and it follows that $\lambda\geq 0$. Thus (a) implies (b).

Now suppose that T is normal with $\sigma(T) \subset [0,\infty)$. Then $f(\lambda) = \sqrt{\lambda}$ is in $C\big(\sigma(T)\big)$ and $S = f(T)$ satisfies $S = S^*$ and $S^2 = T$. Therefore, (b) implies (c), and (c) implies (a) is immediate.

Since $S = R^2$ for $R = \sqrt[4]{T}$, the above argument shows that T has a positive square root. Since $\lambda \mapsto \sqrt{\lambda}$ is uniformly approximated by polynomials on $[0, \|T\|]$, we can approximate S by polynomials in T.

Thus if S' is another positive square root, then S' commutes with T and therefore with S as well. Therefore, $C^*\big(\{I, S, S'\}\big)$ is a commutative C^*-algebra which is isomorphic to $C(\Delta)$ for a compact Hausforff space Δ (by the Abstract Spectral Theorem). But $\hat{S}$ and $\hat{S}'$ are nonnegative functions: for example, $\hat{S}(\Delta) = \sigma(S) \subset [0,\infty)$. Then $\hat{S}^2 = \hat{S'}^2$ implies that $\hat{S} = \hat{S}'$. Therefore, $S = S'$. □

Exercises

E 2.2.1. At the risk of offending any algebraists, view the algebra $\mathbf{C}[x,y]$ of complex polynomials in two variables x and y as the vector space with basis $\{\, x^n y^m : n \text{ and } m \text{ nonnegative integrers}\,\}$ and multiplication compatible with $(x^n y^m)(x^k y^l) = x^{n+k} y^{m+l}$. Then if $f \in \mathbf{C}[x,y]$ and both a and b are elements of an algebra $\mathscr{A}$, then it is clear what we mean by $f(a,b) \in \mathscr{A}$. Suppose now that a is a normal element in a unital C^*-algebra A. Let $\mathscr{A} = \{\, f(a,a^*) : f \in \mathbf{C}[x,y]\,\}$. Show that $C^*(\{\mathbf{1}_A, a\})$ is the closure of $\mathscr{A}$ in A. In particular, conclude that $C^*(\{\mathbf{1}_A, a\})$ is commutative.

E 2.2.2. Extend E 2.2.1 to show that if S is any self-adjoint subset of unital C^*-algebra A such that $a, b \in S$ implies $ab = ba$, then $C^*(S)$ is commutative.

E 2.2.3. Prove Corollary 2.19.

E 2.2.4. Suppose that A is a unital C^*-algebra and that $f : \mathbf{R} \to \mathbf{C}$ is continuous. Show that the map $x \mapsto f(x)$ is a continuous map from $A_{s.a.} = \{\, x \in A : x = x^*\,\}$ to A.

—Answer on page 136

E 2.2.5. As in ([5]), use the maximum modulus theorem to view the disk algebra, $A(D)$, as a Banach subalgebra of $C(\mathbf{T})$.[6] Let $f \in A(D)$ be the identity function: $f(z) = z$ for all $z \in \mathbf{T}$. Show that $\sigma_{C(\mathbf{T})}(f) = \mathbf{T}$, while $\sigma_{A(D)}(f) = \overline{D}$. This shows that, unlike the case of C^*-algebras where we have "spectral permanence," we can have $\sigma_A(b)$ a proper subset of $\sigma_B(b)$ when B is a unital subalgebra of A.

—Answer on page 137

2.3 Positivity

Definition 2.24. A self-adjoint element in a C^*-algebra is called *positive* if $\sigma(a) \subset [0, \infty)$. If a is positive we'll write $a \geq 0$, and we let $A^+ = \{\, a \in A : a \geq 0 \,\}$.

Theorem 2.25 (Spectral Mapping Theorem)**.** *Suppose that a is a normal element in a C^*-algebra. If $f \in C\big(\sigma(a)\big)$, then let $f(a)$ be the image of a under the functional calculus. (Hence, if A is not unital, then we must assume $f(0) = 0$.) Then $\sigma\big(f(a)\big) = f\big(\sigma(a)\big)$.*

Proof. We can assume that A has an identity. Let $\Psi : C\big(\sigma(a)\big) \to B := C^*\big(\{\mathbf{1}, a\}\big)$ be the functional calculus isomorphism (so that $f(a) = \Psi(f)$). By spectral permanence,

$$\sigma_A\big(f(a)\big) = \sigma_B\big(f(a)\big) = \sigma_{C(\sigma(a))}(f) = f\big(\sigma(a)\big). \qquad \square$$

Recall that if A is a C^*-algebra, then we use $\widetilde{A}$ to be A^1 is A does not have an identity and just A if A has an identity.

Lemma 2.26. *Suppose that A is a C^*-algebra.*

(a) *If $a \in A$ is normal, then $a^*a \geq 0$.*
(b) *If $a \in A$ is self-adjoint and $\|a\| \leq 1$, then $a \geq 0$ if and only if $\|\mathbf{1}_A - a\| \leq 1$ (in $\widetilde{A}$).*
(c) *If $a, b \in A^+$, then $a + b \in A^+$.*
(d) *A^+ is closed in A.*

Remark 2.27. Since A^+ is obviously closed under scalar multiplication by nonnegative scalars, part (c) says that A^+ is a *cone*. Thus A^+ is often referred to as the *positive cone in A.*

[5] Cross reference needed here.

[6] Although it is not relevant to the problem, we can put an involution on $C(\mathbf{T})$, $f^*(z) = \overline{f(\bar{z})}$, making $A(D)$ a Banach $*$-subalgebra of $C(T)$. You can then check that neither $C(\mathbf{T})$ nor $A(D)$ is a C^*-algebra with respect to this involution.

Proof. If a is normal, then by Spectral Mapping Theorem, $\sigma(a^*a) = f(\sigma(a))$ where $f(\lambda) = \bar{\lambda}\lambda = |\lambda|^2$. Part (a) follows.

To prove (b), let $D = C^*\big(\{\mathbf{1}, a\}\big) \subset \widetilde{A}$. Let $\Psi : C\big(\sigma(a)\big) \to D$ be the functional calculus map. Since $\|a\| = \rho(a) = \|\hat{a}\|_\infty$,

$$\sigma(a) \subset [-1, 1].$$

But $\|\mathbf{1} - a\| = \|1 - \mathrm{id}\,\|_\infty$ on $\sigma(a)$. Thus $\|\mathbf{1} - a\| \leq 1$ if and only if $\sigma(a) \subset [0, 1]$. This proves (b).

To prove (c), we may as well assume $\|a\| \leq 1$ and $\|b\| \leq 1$. By (b), this means $\|\mathbf{1} - a\| \leq 1$ and $\|\mathbf{1} - b\| \leq 1$. But then $\|\mathbf{1} - \frac{1}{2}(a + b)\| \leq 1$. Therefore $\frac{1}{2}(a + b) \geq 0$ by part (b). Hence $a + b \geq 0$ as required.

Part (d) also follows from part (b). □

We want to verify that in a C^*-algebra a^*a is always positive — so that the hypothesis in part (a) of Lemma 2.26 is unnecessary. But we have to work a bit since we can't use our favorite tool, the functional calculus, directly on not necessarily normal elements.

Lemma 2.28. *Suppose $a, b \in A$. Then $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.*

Proof. We can assume that A has a unit. A little thought should convince that it suffices to prove that $\mathbf{1} - ab \in \mathrm{Inv}(A)$ implies $\mathbf{1} - ba \in \mathrm{Inv}(A)$.

Let $u := (\mathbf{1} - ab)^{-1}$. Then

$$\begin{aligned}(1 + bua)(\mathbf{1} - ba) &= \mathbf{1} - ba + bua - buaba \\ &= \mathbf{1} - b(\mathbf{1} - u + uab)a \\ &= \mathbf{1} - b(\mathbf{1} - u(\mathbf{1} - ab))a \\ &= \mathbf{1}.\end{aligned}$$

A similar computation shows that $(\mathbf{1} - ba)(\mathbf{1} + bua) = \mathbf{1}$. Hence $\mathbf{1} - ba \in \mathrm{Inv}(A)$. □

Lemma 2.29. *If $a \in A$ and $\sigma(a^*a) \subset (-\infty, 0]$, then $a = 0$.*

Proof. Lemma 2.28 implies that $\sigma(aa^*) \subset (-\infty, 0]$. Since $-a^*a$ and $-aa^*$ are in A^+ and since A^+ is a cone, we have

$$\sigma(a^*a + aa^*) \subset (-\infty, 0].$$

Let $a = x + iy$ with $x, y \in A_{s.a.}$, and notice that $a^*a + aa^* = 2x^2 + 2y^2$.

Since x^2 and y^2 are positive (by Lemma 2.26), $2x^2 + 2y^2 = a^*a + aa^*$ is positive and $\sigma(a^*a + aa^*) \subset [0, \infty)$. Thus, $\sigma(a^*a + aa^*) = \{0\}$. Since $a^*a + aa^*$ is self-adjoint, $a^*a + aa^* = 0$. But this forces $x^2 = -y^2$. Therefore $\sigma(x)$ and $\sigma(y)$ are both $\{0\}$, and $x = y = 0$. But then $a = 0$ as claimed. $\square$

Now we have to tools to prove a C^*-version of Theorem 2.23 on page 36.

Theorem 2.30. *Suppose that A is a C^*-algebra and that $a \in A$. Then the following are equivalent.*

(a) $a \geq 0$.

(b) *There is a unique $a^{\frac{1}{2}} \in A^+$ such that $(a^{\frac{1}{2}})^2 = a$.*

(c) $a = b^*b$ *for some* $b \in A$.

Proof. We have (a)$\Longrightarrow$(b) by the functional calculus (with the uniqueness proved just as in Theorem 2.23). Of course, (b)$\Longrightarrow$(c) is immediate.

To see that (c)$\Longrightarrow$(a), start by defining

$$f(t) = \begin{cases} \sqrt{t} & \text{if } t \geq 0 \text{ and} \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad g(t) = \begin{cases} 0 & \text{if } t \geq 0, \text{ and} \\ \sqrt{-t} & \text{otherwise.} \end{cases}$$

Then for all $t \in \mathbf{R}$ we have $f(t)g(t) = 0$ and $f(t)^2 - g(t)^2 = t$. Since f and g both vanish at 0, we get self-adjoint elements of A via

$$u = f(b^*b) \quad \text{and} \quad v = g(b^*b).$$

Then $uv = vu = 0$ and $u^2 - v^2 = b^*b$. But then $vb^*bv = v(u^2 - v^2)v = -v^4$. Thus $\sigma((bv)^*bv) = \sigma(-v^4) \subset (-\infty, 0]$. Thus $-v^4 = 0$. Since v is self-adjoint, this means $v = 0$. But then $b^*b = u^2$, and u^2 is positive by Lemma 2.26. $\square$

Remark 2.31. Note that by spectral permanence, if B is a C^*-subalgebra of A and if $b \in B$ is in A^+, then $b \in B^+$.

Remark 2.32. Note that we can define a partial order on $A_{s.a.}$ by $x \leq y$ if and only if $y - x \geq 0$.

Corollary 2.33. *Suppose that $x \leq y$ in $A_{s.a.}$. Then $axa^* \leq aya^*$ for any $a \in A$.*

Proof. Since $y - x \geq 0$, there is a $b \in A$ such that $y - x = bb^*$. But then $aya^* - axa^* = abb^*a^* = (ab)(ab)^* \geq 0$. $\square$

Lemma 2.34. *Suppose that A is a unital C^*-algebra and that $a \in A^+$. Then $a \leq \mathbf{1}_A$ if and only if $\|a\| \leq 1$.*

Proof. We have $\sigma(a) \subset [0, \|a\|]$. But by the functional calculus, $\mathbf{1}_A - a \geq 0$ if and only if $1 - \lambda \geq 0$ for all $\lambda \in \sigma(a)$. □

Lemma 2.35. *Suppose that A is a unital C^*-algebra and that $x, y \in \operatorname{Inv}(A)$ are such that $0 \leq x \leq y$. Then $0 \leq y^{-1} \leq x^{-1}$.*

Remark 2.36. This would follow easily from the functional calculus if x and y commuted.

Proof. By assumption, $\sigma(x)$ and $\sigma(y)$ are both contained in $(0, \infty)$. Thus $x^{\pm\frac{1}{2}}$ and $y^{\pm\frac{1}{2}}$ are well defined elements of A^+. Using Corollary 2.33 and Lemma 2.34, we have

$$\begin{aligned} 0 \leq x \leq y &\Longrightarrow 0 \leq y^{-\frac{1}{2}} x y^{-\frac{1}{2}} \leq \mathbf{1}_A \\ &\Longrightarrow \|y^{-\frac{1}{2}} x y^{-\frac{1}{2}}\| \leq 1 \\ &\Longrightarrow \|y^{-\frac{1}{2}} x^{\frac{1}{2}}\|^2 \leq 1 \\ &\Longrightarrow \|x^{\frac{1}{2}} y^{-1} x^{\frac{1}{2}}\| \leq 1 \\ &\Longrightarrow 0 \leq x^{\frac{1}{2}} y^{-1} x^{\frac{1}{2}} \leq \mathbf{1}_A \\ &\Longrightarrow 0 \leq y^{-1} \leq x^{-1}. \end{aligned}$$
□

Exercises

E 2.3.1. Suppose that a and b are elements in a C^*-algebra A and that $0 \leq a \leq b$. Show that $\|a\| \leq \|b\|$. What happens if we drop the assumption that $0 \leq a$? (Hint: use Lemma 2.34 on the preceding page.)

E 2.3.2. Suppose that a and b are elements in a C^*-algebra A. Show that $b^* a^* a b \leq \|a\|^2 b^* b$. (Hint: Apply Lemma 2.34 to $a^* a$ in $\widetilde{A}$.)

E 2.3.3. Suppose a is a self-adjoint element of a C^*-algebra A. Show that $\|a\| \leq 1$ if and only if $\mathbf{1} - a \geq 0$ and $\mathbf{1} + a \geq 0$ in $\widetilde{A}$.

E 2.3.4. Suppose that U is an bounded operator on a complex Hilbert space $\mathcal{H}$. Show that the following are equivalent.

(a) U is isometric on $\ker(U)^\perp$.

(b) $UU^*U = U$.

(c) UU^* is a projection.[7]

(d) U^*U is a projection.

An operator in $B(\mathcal{H})$ satisfying (a), and hence (a)–(d), is called a *partial isometry* on $\mathcal{H}$. The reason for this terminology ought to be clear from part (a).

Conclude that if U is a partial isometry, then UU^* is the projection on the (necessarily closed) range of U, that U^*U is the projection on the $\ker(U)^\perp$, and that U^* is also a partial isometry.

(Hint: Replacing U by U^*, we see that (b)$\Longleftrightarrow$(c) implies (b)$\Longleftrightarrow$(c)$\Longleftrightarrow$(d). Then use (b)–(d) to prove (a). To prove (c)$\Longrightarrow$(b), consider $(UU^*U - U)(UU^*U - U)^*$.)

—Answer on page 137

E 2.3.5. Suppose that P and Q are projections in $B(H)$. We say that $P \perp Q$ if $P(H) \perp Q(H)$ and that $P \le Q$ if $P(H) \subset Q(H)$.

(a) Show that the following are equivalent.

(i) $P \perp Q$.

(ii) $PQ = QP = 0$.

(iii) $P + Q$ is a projection.

(b) Show that the following are equivalent.

(i) $P \le Q$.

(ii) $PQ = QP = P$.

(iii) $Q - P$ is a projection.

(Hint: Note that PQP is a positive operator. Also $PQP = PQ(PQ)^*$ so that $PQP = 0$ if and only if $PQ = QP = 0$.)

[7] A a bounded operator P on a complex Hilbert space $\mathcal{H}$ is called a *projection* if $P = P^* = P^2$. The term *orthogonal projection* or *self-adjoint projection* is, perhaps, more accurate. Note that $\mathcal{M} = P(\mathcal{H})$ is a *closed* subspace of $\mathcal{H}$ and that P is the usual projection with respect to the direct sum decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. However, since we are only interested in these sorts of projections, we will settle for the undecorated term "projection."

2.4 Approximate Identities

Many C^*-algebras of interest do not have identities and it is not always convenient or appropriate to pass to the unitization. One of the results in this section is that C^*-algebras always almost have an identity. For example, if $A = C_0(\mathbf{R})$, then we can let e_n be the continuous function which is identically 1 on $[-n, n]$, identically zero on $(-\infty, -n-1] \cup [n+1, \infty)$ and linear elsewhere. Then for any $f \in A$, $e_n f \to f$ in norm. In general, we make the following definition.

Definition 2.37. A net $\{e_\lambda\}_{\lambda\in\Lambda}$ in a C^*-algebra A is called an *approximate identity* for A if

AI-1. $e_\lambda \geq 0$ for all λ,

AI-2. $\|e_\lambda\| \leq 1$ for all λ,

AI-3. $\lambda \leq \mu$ implies $e_\lambda \leq e_\mu$ and

AI-4. for all $a \in A$, $\lim_\lambda e_\lambda a = a = \lim_\lambda a e_\lambda$.

Of course, the family $\{e_n\}_{n=1}^\infty \subset C_0(\mathbf{R})$ described above is an example of an approximate identity. In the next result, we show that every C^*-algebra has an approximate identity. In fact, we prove a slightly stronger result for ideals that will be very useful.

Theorem 2.38. *Every C^*-algebra has an approximate identity. More generally if J is a not necessarily closed two-sided ideal in A, then there is a net $\{e_\lambda\}_{\lambda\in\Lambda}$ satisfying* AI-1, AI-2 *and* AI-3 *of Definition 2.37, and such that* AI-4 *holds for all $a \in \overline{J}$.*

Proof. Simple computations show that J will still be an ideal in $\widetilde{A}$, so we can assume A has an identity. Let Λ be the collection of finite subsets of $J_{s.a.}$ directed by containment.[8] Define $f_n : [0, \infty) \to [0, 1)$ by

$$f_n(t) = \frac{nt}{1+nt}.$$

Notice that if $a \in A^+ \cap J$, then

$$f_n(a) = na(1+na)^{-1} \in J$$

since $na \in J$. Thus, if $\lambda = \{x_1, \ldots, x_n\} \in \Lambda$, then $x_1^2 + \cdots + x_n^2 \in A^+$ (by Lemma 2.26. Then we can define

$$e_\lambda = f_n(x_1^2 + \cdots + x_n^2).$$

[8] Since $a \in J$ implies $a^*a \in J$, Λ is nonempty.

Since f_n is nonnegative and bounded by 1, we have $e_\lambda \geq 0$ and $\|e_\lambda\| \leq 1$. Thus conditions AI-1 and AI-2 are clearly satisfied.

Now suppose that $\lambda \leq \mu$ in Λ. Then if $\lambda = \{x_1, \dots, x_m\}$, we must have $\mu = \{x_1, \dots, x_m, \dots, x_n\}$ with $m \leq n$. To show that $e_\lambda \leq e_\mu$ it will suffice to see that

$$\mathbf{1} - e_\mu \leq \mathbf{1} - e_\lambda. \tag{2.3}$$

Since $1 - f_k(t) = 1 - \frac{kt}{1+kt} = \frac{1}{1+kt}$, to show (2.3) it will suffice to see that

$$\big(\mathbf{1} + n(x_1^2 + \cdots + x_n^2)\big)^{-1} \leq \big(\mathbf{1} + m(x_1^2 + \cdots + x_m^2)\big)^{-1}. \tag{2.4}$$

But (2.4) follows from the observation that

$$\mathbf{1} + m(x_1^2 + \cdots + x_m^2) \leq \mathbf{1} + n(x_1^2 + \cdots + x_n^2)$$

and Lemma 2.35 on page 41. Thus, $\{e_\lambda\}_{\lambda \in \Lambda}$ satisfies AI-1, AI-2 and AI-3.

To establish AI-4, we start by supposing that $a \in J_{s.a.}$. Then $\{a\} \in \Lambda$. Fix $\lambda_0 = \{x_1, \dots, x_m\} \geq \{a\}$ (so that some $x_i = a$). Then if $\lambda \geq \lambda_0$ we must have $\lambda = \{x_1, \dots, x_n\}$ with $n \geq m$. But $a^2 \leq x_1^2 + \cdots + x_n^2$ implies that

$$(\mathbf{1} - e_\lambda)a^2(\mathbf{1} - e_\lambda) \leq (\mathbf{1} - e_\lambda)(x_1^2 + \cdots + x_n^2)(\mathbf{1} - e_\lambda). \tag{2.5}$$

Let $g_n(t) = (1 - f_n(t))t(1 - f_n(t)) = \frac{t}{(1+nt)^2}$, and notice that the right-hand side of (2.5) is $g_n(x_1^2 + \cdots + x_n^2)$. Thus we have

$$(a - ae_\lambda)^*(a - ae_\lambda) \leq g_n(x_1^2 + \cdots + x_n^2).$$

Since a little calculus shows that $\|g_n\|_\infty \leq \frac{1}{4n}$ (that maximum occurs at $t = \frac{1}{n}$), we have $\|g_n(x_1^2 + \cdots + x_n^2)\| \leq \frac{1}{4n}$. Now we can apply Lemma 2.34 on page 40 to conclude that

$$g_n(x_1^2 + \cdots + x_n^2) \leq \frac{1}{4n}\mathbf{1}_A,$$

and that for any $\lambda \geq \lambda_0$

$$\|a - ae_\lambda\|^2 \leq \frac{1}{4n} \leq \frac{1}{4m}.$$

Since m is arbitrary, $\lim_\lambda ae_\lambda = a$ for any $a \in J_{s.a.}$.

If $a \in J$, then

$$\begin{aligned} \|a - ae_\lambda\|^2 &= \|a(\mathbf{1} - e_\lambda)\|^2 \\ &= \|(\mathbf{1} - e_\lambda)a^*a(\mathbf{1} - e_\lambda)\| \\ &\leq \|\mathbf{1} - e_\lambda\|\|a^*a - a^*ae_\lambda\| \\ &\leq \|a^*a - a^*ae_\lambda\| \end{aligned}$$

which tends to zero since $a^*a \in J_{s.a.}$.

On the other hand,

$$\begin{aligned} \|a - e_\lambda a\|^2 &= \|(\mathbf{1} - e_\lambda)a\|^2 \\ &= \|(\mathbf{1} - e_\lambda)aa^*(\mathbf{1} - e_\lambda)\| \\ &\leq \|aa^* - aa^*e_\lambda\| \end{aligned}$$

which also tends to 0.

This proves AI-4 for $a \in J$. We easily extend this to $a \in \overline{J}$ using the triangle inequality and AI-2. □

Corollary 2.39. *Suppose that A is a separable C^*-algebra. Then Theorem 2.38 holds with $\{e_\lambda\}_{\lambda\in\Lambda}$ replaced by a sequence $\{e_n\}_{n=1}^\infty$. In particular, every separable C^*-algebra has an approximate identity which is a sequence.*

Proof. Since any subset of a separable metric space is separable, $J_{s.a.}$ is separable. Suppose $D = \{x_i\}_{i=1}^\infty$ is a dense sequence in $J_{s.a.}$. Let $e_n := e_{\{x_1,\dots,x_n\}}$. Properties AI-1, AI-2 and AI-3 are clear. Showing AI-4 holds is an exercise.[9] □

Corollary 2.40. *Every closed two-sided ideal in a C^*-algebra is self-adjoint and therefore a C^*-subalgebra.*

Proof. Let J be an ideal in A. Let $\{e_\lambda\}$ be an approximate identity for J. If $a \in J$, then $ae_\lambda \to a$. Since the $*$-operation is isometric, $e_\lambda a^* \to a^*$. Since each $e_\lambda a^* \in J$, we must have $a^* \in \overline{J}$. □

Theorem 2.41. *Suppose that J is an ideal in a C^*-algebra A. Let $q : A \to A/J$ be the quotient map. Then*

$$q(a)^* := q(a^*) \tag{2.6}$$

is a well-defined involution on A/J making it into a C^*-algebra in the quotient norm.

[9] Reference

We already know, from Theorem 1.15 on page 6, that A/J is a Banach algebra with respect to the quotient norm. Furthermore, (2.6) is clearly well-defined in view of Corollary 2.40. All that remains is to show that the quotient norm is isometric (so that A/J is a Banach $*$-algebra) that it satisfies the C^*-identity:

$$\|q(a)^*q(a)\| = \|q(a)\|^2.$$

To do this, we need the following result which is worth separating out as a lemma.

Lemma 2.42. *Suppose that J is an ideal in A and that $\{e_\lambda\}$ is an approximate identity for J. Then for all $a \in A$, we have*

$$\|q(a)\| = \lim_\lambda \|a - ae_\lambda\|.$$

Proof. Since $y \in J$ implies that $\|y - ye_\lambda\| \to 0$, it follows that for all $y \in J$

$$\begin{aligned}\limsup_\lambda \|a - ae_\lambda\| &= \limsup_\lambda \|a - ae_\lambda + y - ye_\lambda\| \\ &= \limsup_\lambda \|(\mathbf{1} - e_\lambda)(a + y)\| \\ &\le \|a + y\|.\end{aligned}$$

Since $y \in J$ is arbitrary,

$$\limsup_\lambda \|a - ae_\lambda\| \le \|q(a)\|.$$

On the other hand, $ae_\lambda \in J$ for all λ, so $\|a - ae_\lambda\| \ge \|q(a)\|$. Therefore,

$$\liminf_\lambda \|a - ae_\lambda| \ge \|q(a)\|.$$

It follows that $\lim_\lambda \|a - ae_\lambda\|$ exists and equals $\|q(a)\|$. $\square$

Proof of Theorem 2.41. Using the above result,

$$\begin{aligned}\|q(a)\|^2 &= \lim_\lambda \|a - ae_\lambda\|^2 \\ &= \lim_\lambda \|(a - ae_\lambda)^*(a - ae_\lambda)\| \\ &= \lim_\lambda \|(\mathbf{1} - e_\lambda)a^*a(\mathbf{1} - e_\lambda)\| \\ &\le \limsup_\lambda \|a^*a(\mathbf{1} - e_\lambda)\| \\ &= \|q(a^*a)\| = \|q(a)^*q(a)\|.\end{aligned}$$

It now follows from HW[10] that $\|q(a)^*\| = \|q(a)\|$ and that $\|q(a)^*q(a)\| = \|q(a)\|^2$. $\square$

[10] Reference

EXERCISES

E 2.4.1. In this question, ideal always means 'closed two-sided ideal.'

(a) Suppose that I and J are ideals in a C^*-algebra A. Show that IJ — defined to be the closed linear span of products from I and J — equals $I \bigcap J$.

(b) Suppose that J is an ideal in a C^*-algebra A, and that I is an ideal in J. Show that I is an ideal in A.

—Answer on page 138

E 2.4.2. Prove Corollary 2.39 on page 45: Show that every separable C^*-algebra contains a sequence which is an approximate identity. (Recall that we showed in the proof of Theorem 2.38 that if $x \in A_{\text{s.a.}}$, and if $x \in \{\, x_1, \dots, x_n \,\} = \lambda$, then $\|x - xe_\lambda\|^2 < 1/4n$.)

—Answer on page 138

2.5 Homomorphisms

Theorem 2.43. *Suppose that A is a Banach $*$-algebra and that B is a C^*-algebra. Then every $*$-homomorphism $\varphi : A \to B$ is continuous and satisfies $\|\varphi\| \le 1$. If A is a C^*-algebra, then $\varphi(A)$ is a C^*-subalgebra of B and φ induces an isometric $*$-isomorphism of $A/\ker\varphi$ onto $\varphi(A)$.*

Proof. To begin, assume that A and B are both unital and that $\varphi(\mathbf{1}_A) = \mathbf{1}_B$. Then $a \in \operatorname{Inv}(A)$ implies $\varphi(a) \in \operatorname{Inv}(B)$. Thus

$$\sigma_B\big(\Phi(a)\big) \subset \sigma_A(a). \tag{2.7}$$

Recall that if $b \in B_{s.a.}$, then $\|b\| = \rho(b)$. Thus for any $a \in A$, we have

$$\|\varphi(a)\|^2 = \|\varphi(a^*a)\| = \rho\big(\varphi(a^*a)\big) \le \rho(a^*a) \le \|a^*a\| \le \|a\|^2, \tag{2.8}$$

where the first inequality comes from (2.7). This shows that $\|\varphi\| \le 1$.

Now assume in addition to φ being unital, A is a C^*-algebra with $\ker\varphi = \{0\}$. Then to see that φ is isometric, it suffices to see that if $a = a^*$, then

$$\sigma_B\big(\Phi(a)\big) = \sigma_A(a).$$

(Then all the inequalities in (2.8) are equalities.) But if this fails, then $\sigma_B\big(\varphi(a)\big)$ is a proper closed subset of $\sigma_A(a)$. Here there is a nonzero *real-valued* continuous function on $\sigma_A(a)$ such that $f(\lambda) = 0$ for all $\lambda \in \sigma_B\big(\varphi(a)\big)$. Since f is real-valued, there are polynomials $p_n \to f$ uniformly on $\sigma_A(a)$. But if p is a polynomial, then

$$\varphi(p(a)) = p\big(\varphi(a)\big).$$

But we must have $p_n(a) \to f(a)$. Since φ is continuous, $p_n\big(\varphi(a)\big) = \varphi\big(p_n(a)\big) \to \varphi\big(f(a)\big)$. On the other hand, we must also have $p_n\big(\varphi(a)\big) \to f\big(\varphi(a)\big)$. Thus,

$$f\big(\varphi(a)\big) = \varphi\big(f(a)\big).$$

However, since f vanishes on $\sigma_B\big(\varphi(a)\big)$, we have $f\big(\varphi(a)\big) = \varphi\big(f(a)\big) = 0$. Since φ is injective, this means $f(a) = 0$. But $f \neq 0$ on $\sigma_A(a)$, so this is a contradiction.

We conclude that φ is isometric. But then $\varphi(A)$ is complete, and hence closed.

Now, still in the case φ is unital and that A is a C^*-algebra, consider the situation where $J := \ker\varphi \neq \{0\}$. Then J is an ideal in A and φ induces a injective unital $*$-homomorphism $\dot\varphi : A/J \to B$ given by $\dot\varphi(a + J) = \varphi(a)$. But the above argument shows that $\dot\varphi$ is isometric and that $\varphi(A) = \dot\varphi(A/J)$ is a C^*-subalgebra.

If we only assume that A is unital, then $\varphi(\mathbf{1}_A)$ is a unit for $\varphi(A)$ and hence also for $C := \overline{\varphi(A)}$. This puts us back in the case φ is unital, and the theorem follows whenever A has a unit.

If A is not unital, define $\varphi^1 : A^1 \to B^1$ by $\varphi^1((a,\lambda)) = \big(\varphi(a),\lambda\big)$. Then φ^1 is a $*$-homomorphism. If A is a Banach $*$-algebra, then so is A^1 and $\|\varphi^1\| \leq 1$. Therefore $\|\varphi\| \leq 1$. If A is a C^*-algebra, then we can assume A^1 is as well. If we view A as a subalgebra of A^1, then notice that

$$\ker\varphi = \ker\varphi^1.$$

Hence φ^1, and therefore φ as well, is isometric if $\ker\varphi = \{0\}$. If $\ker\varphi = J \neq \{0\}$, then

$$(\dot\varphi)^1 : (A/J)^1 \to B^1$$

is isometric. Therefore $\dot\varphi : A/J \to B$ is as well. □

To Do: Make up a problem about $M_n \otimes A$ where we can use the magic of automatic continuity.

Exercises

E 2.5.1. Show that a $*$-algebra admits at most one norm making it into a C^*-algebra.

Chapter 3

Representations

3.1 Representations

Definition 3.1. A *representation* of a C^*-algebra A is a $*$-homomorphism π from A into the bounded operators $B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. We say that π is *nondegenerate* if

$$[\pi(A)\mathcal{H}] := \overline{\operatorname{span}}\{\, \pi(a)h : a \in A \text{ and } h \in \mathcal{H} \,\} = \mathcal{H}.$$

A subspace $V \subset \mathcal{H}$ is called *invariant* for π if $\pi(a)V \subset V$ for all $a \in A$. We say that V is a *cyclic subspace* for π if there is a $h \in V$, called a *cyclic vector*, such that

$$[\pi(A)h] := \overline{\operatorname{span}}\{\, \pi(a)h : a \in A \,\} = V.$$

We say that π is a *cyclic representation* if $\mathcal{H}$ is a cyclic subspace for π.

A representation $\rho : A \to B(\mathcal{V})$ is *equivalent* to π if there is a unitary $U : \mathcal{H} \to \mathcal{V}$ so that

$$\rho(a) = U\pi(a)U^* \quad \text{for all } a \in A.$$

In this case we write $\pi \sim \rho$.

Remark 3.2. Note that $\pi : A \to B(\mathcal{H})$ is nondegenerate exactly when

$$\{\, h \in \mathcal{H} : \pi(a)h = 0 \text{ for all } a \in A \,\} = \{0\}.$$

Remark 3.3. If V is an invariant subspace for π, then so is $V^\perp$.

Definition 3.4. A nonzero representation $\pi : A \to B(\mathcal{H})$ is called *irreducible* if π has no nontrivial closed invariant subspaces.

Definition 3.5. If $\mathscr{S}$ is a subset of $B(\mathcal{H})$, then the *commutant* of $\mathscr{S}$ is

$$\mathscr{S}' := \{\, T \in B(\mathcal{H}) : ST = TS \text{ for all } S \in \mathscr{S}. \,\}$$

Note that if $\mathscr{S}$ is self-adjoint, then $\mathscr{S}'$ is always unital C^*-subalgebra of $B(\mathcal{H})$. (In fact, $\mathscr{S}'$ is closed in the weak operator topology. But that is another story.) In this section, we are interested in the commutant of a representation:

$$\pi(A)' = \{\, T \in B(\mathcal{H}) : T\pi(a) = \pi(a)T \text{ for all } a \in A \,\}.$$

Proposition 3.6. *Suppose that $\pi : A \to B(\mathcal{H})$ is a representation. Then the following statements are equivalent.*

(a) *π is irreducible.*
(b) *$\pi(A)' = \mathbf{C}I_{\mathcal{H}}$.*
(c) *$\pi(A)'$ contains no nontrivial projections.*
(d) *Every $h \in \mathcal{H} \setminus \{0\}$ is cyclic for π.*

Proof. Note that if P is a projection in $B(\mathcal{H})$, then $h \in \mathcal{H}$ is in the space of P if and only if $Ph = h$. It follows that P is in $\pi(A)'$ if and only if the space of P is invariant. Hence (a) and (c) are equivalent. Also if $h \in \mathcal{H} \setminus \{0\}$, then $\overline{\{\, \pi(a)h : a \in A \,\}}$ is an invariant subspace. It follows that (a) and (d) are equivalent. Since (b)$\Longrightarrow$(c) is clear, it suffice to prove (a)$\Longrightarrow$(b). However, if $\pi(A)'$ is nontivial, then it contains a nontrivial self-adjoint operator T. Since $C^*(\{\, T, I \,\}) \subset \pi(A)'$ and $\sigma(T)$ contains at least two points, there are nonzero operators $A, B \in C^*(\{\, T, I \,\}) \subset \pi(A)'$ such that $AB = 0$. Then the range $B(\mathcal{H})$ is invariant and in the kernel of A. Hence its closure in a nontrivial invariant subspace. This completes the proof. □

Definition 3.7. If A is a C^*-algebra, then the set of equivalence classes of irreducible representations is called the *spectrum* of A and is denoted by $\hat{A}$.

Remark 3.8. If you're keeping score, then you'll recall that we also defined the spectrum of a commutative Banach algebra A to be the set $\Delta(A)$ of nonzero complex homomorphisms $h : A \to \mathbf{C}$. But if A is a commutative C^*-algebra, and if we make the obvious identification of $\mathbf{C}$ with the bounded operators on a one-dimensional Hilbert space, it is clear that we can think of nonzero complex homomorphisms as irreducible one-dimensional representations of A. Furthermore, every one-dimensional representation[1] is a complex homomorphism and any two are equivalent if and only if they are equal.

In view of the above, our dual use of the term "spectrum" is justified by the next result.

[1]The dimension of a representation is the dimension of the Hilbert space it acts on.

Theorem 3.9. *If A is a commutative C^*-algebra, then every irreducible representation is one-dimensional. Therefore, for commutative C^*-algebras, we can identify $\hat{A}$ with $\Delta(A)$.*

Proof. Suppose that $\pi : A \to B(\mathcal{H})$ is an irreducible representation. If A is commutative, then $\pi(A)$ is a commutative C^*-algebra (by Theorem 2.43 on page 47), and hence isomorphic to $C_0(\Delta)$. If Δ is reduced to a point, we're done (because the only way the scalars can act irreducibly is if the Hilbert space is one-dimensional).

Otherwise, $\pi(A)$ must contain a proper nonzero ideal J as well as a self-adjoint operator T such that $JT = \{0\}$.[2] Since J is an ideal in $\pi(A)$,

$$V := [J\mathcal{H}] = \overline{\operatorname{span}}\{\, Sh : S \in J \text{ and } h \in \mathcal{H} \,\}$$

is a nonzero invariant subspace for π. However, for all $h, k \in H$ and $S \in J$,

$$(Tk \mid Sh) = (k \mid TSh) = 0.$$

Since $T \neq 0$, it follows that $V^\perp \neq \{0\}$. This contradicts the irreducibility of π.

Therefore Δ is a single point and we're done. □

Remark 3.10. Speaking a bit informally, it might be said that a lot of basic C^*-theory is devoted to exploring to what extent the Gelfand theory for commutative Banach algebras — which after all, works as well as can be hoped in the C^*-category — can be extended to noncommutative C^*-algebras by letting irreducible representations play the role of complex homomorphisms.

To Do: Work in the exercise showing that

$$0 \longrightarrow C_0(U) \longrightarrow C_0(X) \longrightarrow C_0(F) \longrightarrow 0$$

and make a link to irreducible representations and Remark 3.10.

[2] It suffices to show that $C_0(\Delta)$ has this property if Δ has at least two distinct points x and y. But x and y have disjoint neighborhoods U and V. Let J be the ideal of functions vanishing off U and let f be a real-valued function with support in V and such that $f(y) = 1$.

Exercises

E 3.1.1. Prove the statement made in Remark 3.2 on page 49.

E 3.1.2. Prove the statement made in Remark 3.3 on page 49.

E 3.1.3. Suppose that π is a non-degenerate representation of A on $\mathcal{H}$. Let $\{ e_\lambda \}$ be an approximate identity for A. Show that $\pi(e_\lambda)$ converges to I in the strong operator topology; that is, prove that $\lim_\lambda \pi(e_\lambda)h = h$ for all $h \in \mathcal{H}$. Conclude that $S = \{ \pi(x)h : x \in A \text{ and } h \in \mathcal{H} \}$ is dense in $\mathcal{H}$. (The point is that *a priori* all we are given is that S *spans* a dense subset of $\mathcal{H}$.)

—Answer on page 138

E 3.1.4. Prove Proposition 3.6. (Suggestions. Observe that $\pi(A)'$ is a C^*-algebra. If $A \in \pi(A)'_{\text{s.a.}}$ and $A \neq \alpha I$ for some $\alpha \in \mathbf{C}$, then use the Spectral Theorem to produce nonzero operators $B_1, B_2 \in \pi(A)'$ with $B_1 B_2 = B_2 B_1 = 0$. Observe that the closure of the range of B_1 is a non-trivial invariant subspace for π.)

3.2 Representations and Ideals

Proposition 3.11. *Suppose that J is an ideal in A and that ρ is a nondegenerate representation of J on $\mathcal{H}$. Then there is a unique representation $\bar\rho$ of A on $\mathcal{H}$ extending ρ. If ρ and ρ' are equivalent representations of J, then there extensions, $\bar\rho$ and $\bar\rho'$ are equivalent representations of A.*

Proof. Suppose that $b_1, \dots, b_n \in J$ and $h_1, \dots, h_n \in \mathcal{H}$. Then for any $a \in A$, I claim that

$$\Big\| \sum_{i=1}^n \rho(ab_i)h_i \Big\| \le \|a\| \Big\| \sum_{i=1}^n \rho(b_i)h_i \Big\|. \tag{3.1}$$

To see this, let $\{e_\lambda\}$ be an approximate identity for J. Then

$$\begin{aligned}\Big\|\sum_{i=1}^n \rho(ab_i)h_i\Big\| &= \lim_\lambda \Big\|\sum_{i=1}^n \rho(ae_\lambda b_i)h_i\Big\| \\ &= \lim_\lambda \Big\|\rho(ae_\lambda)\sum_{i=1}^n \rho(b_i)h_i\Big\| \\ &\le \limsup_\lambda \|\rho(ae_\lambda)\|\Big\|\sum_{i=1}^n \rho(b_i)h_i\Big\| \\ &\le \|a\|\Big\|\sum_{i=1}^n \rho(b_i)h_i\Big\|\end{aligned}$$

which establishes (3.1).

Let $\mathcal{H}_0 := \operatorname{span}\{\, \rho(b)h : b \in J \text{ and } h \in \mathcal{H}\,\}$. Notice that (3.1) implies that if

$$\sum_{i=1}^k \rho(b_i)h_i = 0 \quad \text{then we also have} \quad \sum_{i=1}^k \rho(ab_i)h_i = 0.$$

This means that for each $a \in A$, there is a well defined map, $\bar\rho(a)$, on $\mathcal{H}_0$ that sends

$$\sum_{i=1}^n \rho(b_i)h_i \mapsto \sum_{i=1}^n \rho(ab_i)h_i.$$

Since $\mathcal{H}_0$ is dense and since (3.1) also implies that $\bar\rho(a)$ is bounded, it follows that $\bar\rho(a)$ extends to a bounded operator, that we still call $\bar\rho(a)$, on $\mathcal{H}$ with norm at most $\|a\|$.

Then it is routine to check that $a \mapsto \bar\rho(a)$ is a representation extending ρ. If σ were any other representation extending ρ, then $\sigma(a)\rho(b)h = \rho(ab)h = \bar\rho(a)\rho(b)h$. Since that $\rho(b)h$ span a dense subspace, $\sigma = \bar\rho$.

If $\rho \sim \rho'$, then there is a unitary U such that

$$\rho'(b) = U\rho(b)U^* \quad \text{for all } b \in J.$$

But then,

$$\begin{aligned}U\bar\rho(a)U^*\rho'(b)h &= U\bar\rho(a)\rho(b)U^*h \\ &= U\rho(ab)U^*h \\ &= \rho'(ab)h \\ &= \bar\rho'(a)\rho'(b)h.\end{aligned}$$

Since the $\rho'(b)h$ span a dense subspace, we have $\bar{\rho}'(a) = U\bar{\rho}(a)U^*$ for all $a \in A$, and $\bar{\rho}' \sim \bar{\rho}$ as claimed. □

It is fairly clear that if ρ is an irreducible representation of J, then its extension, $\bar{\rho}$, must be an irreducible representation of A. For example, any invariant subspace for $\bar{\rho}$ is certainly invariant for ρ. Our next result considers the other direction: when is the restriction to J of an irreducible representation of A irreducible. The answer turns out to be "whenever it obviously isn't".

Theorem 3.12. *Suppose that J is an ideal in a C^*-algebra A. If π is an irreducible representation of A and if $\pi(J) \neq \{0\}$, then $\pi|_J$ is irreducible. Conversely, every irreducible representation of J extends uniquely to an irreducible representation of A. If two such representations of J are equivalent, then so are their extensions.*

Proof. Suppose π is an irreducible representation of A and that $\pi(J) \neq \{0\}$. To see that $\pi|_J$ is irreducible, it will suffice to see that $[\pi(J)h] = \mathcal{H}$ for all $h \in \mathcal{H} \setminus \{0\}$. However, $[\pi(J)h]$ is invariant for π, so it must either be $\{0\}$ or all of $\mathcal{H}$.

Suppose that $[\pi(J)h] = \{0\}$. Then $(h \mid \pi(a)k) = 0$ for all $a \in J$ and $k \in \mathcal{H}$. That is, $h \perp [\pi(J)\mathcal{H}]$. But $[\pi(J)\mathcal{H}]$ is invariant for π and nonzero by assumption. This forces $h = 0$. Hence $\pi|_J$ is irreducible.

The rest follows easily from Proposition 3.11. □

Remark 3.13. Suppose that J is an ideal in A. If π is representation of A that kills J — that is, $J \subset \ker \pi$ — then the induced map $\dot{\pi} : A/J \to B(\mathcal{H}_\pi)$ is a representation of A/J which is irreducible if and only if π is. Conversely, if $\dot{\pi}$ is an irreducible representation of A/J, and if $q : A \to A/J$ is the quotient map, then $\pi := \dot{\pi} \circ q$ is an irreducible representation of A. It is also straightforward to see that two representations $\dot{\pi}$ and $\dot{\rho}$ are equivalent representations of A/J if and only if $\dot{\pi} \circ q$ and $\dot{\rho} \circ q$ are equivalent representations of A. These observations together with Theorem 3.12 imply that we can identify $\hat{A}$ with the disjoint union of $\hat{J}$ and $(A/J)^\wedge$.

Exercises

E 3.2.1. Let J be an ideal in a C*-algebra A. We call J a *primitive* ideal if $J = \ker \pi$ for some irreducible representation π of A. On the other hand, J is called *prime* if whenever I_1 and I_2 are ideal in A such that $I_1 I_2 \subseteq J$, then either $I_1 \subset J$ or $I_2 \subset J$.

Show that every primitive ideal in a C*-algebra is prime[3]. (Suggestion: If $I \not\subseteq J$, then if π is an irreducible representation of A with $\ker\pi = J$, $\pi|_I$ is irreducible and $[\pi(I)\xi] = \mathcal{H}$ for all $\xi \in \mathcal{H} \setminus \{0\}$.)

E 3.2.2. Let $\operatorname{Prim}(A)$ be the set of primitive ideals of a C*-algebra A. If $S \subseteq \operatorname{Prim}(A)$, then define $\ker(S) = \bigcap_{P\in S} P$ (with $\ker(\emptyset) = A$). Also if I is an ideal in A, then define $\operatorname{hull}(I) = \{\, P \in \operatorname{Prim}(A) : I \subseteq P \,\}$. Finally, for each $S \subset \operatorname{Prim}(A)$, set $\overline{S} = \operatorname{hull}\bigl(\ker(S)\bigr)$.

(a) Show that if $R_1, R_2 \subset \operatorname{Prim}(A)$, then $\overline{R_1 \cup R_2} = \overline{R_1} \cup \overline{R_2}$.

(b) Show that if $R_\lambda \in \operatorname{Prim}(A)$ for all $\lambda \in \Lambda$, then $\overline{\bigcap_{\lambda\in\Lambda} \overline{R_\lambda}} = \bigcap_{\lambda\in\Lambda} \overline{R_\lambda}$.

(c) Conclude that there is a unique topology on $\operatorname{Prim}(A)$ so that $\{\, \overline{S} : S \subseteq \operatorname{Prim}(A) \,\}$ are the closed subsets.

This topology is called the *hull-kernel* or *Jacobson* topology.

E 3.2.3. Consider the C*-algebras

(a) $A = C_0(X)$, with X locally compact Hausdorff.

(b) $B = C\bigl([0,1], M_2\bigr)$, the set of continuous functions from $[0,1]$ to M_2 with the sup-norm and pointwise operations.

(c) $C = \{\, f \in B : f(0) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right), \alpha \in \mathbf{C} \,\}$.

(d) $D = \{\, f \in B : f(0) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right), \alpha, \beta \in \mathbf{C} \,\}$.

For each of the above discuss the primitive ideal space and its topology. For example, show that $\operatorname{Prim}(A)$ is homeomorphic to X. Notice that all of the above are CCR.[4]

E 3.2.4. Let A be a C^*-algebra without unit. Then $\widetilde{A}$ is the smallest C^*-algebra with unit containing A as an ideal. It has become apparent in the last few years, that it is convenient to work with the "largest" such algebra (in a sense to be made precise below). For motivation, suppose that A sits in B as an ideal. Then each $b \in B$ defines a pair of operators $L, R \in \mathscr{B}(A)$ defined by $L(a) = ba$ and $R(a) = ab$. Note that for all $a, c \in A$,

$$(1) \quad L(ac) = L(a)c, \qquad (2) \quad R(ac) = aR(c), \qquad (3) \quad aL(c) = R(a)c.$$

[3] If A is separable, the converse holds [RW98, Theorem A.49]. It has just recently been discovered that the converse can fail without the separable assumption [Wea03].

[4] A C^*-algebra A is called CCR if $\pi(A) = \mathcal{K}(\mathcal{H}_\pi)$ for all irreducible representations π of A.

Define a *multiplier* or *double centralizer* on A to be a pair (L, R) of operators on A satisfying conditions (1), (2), and (3) above. Let $M(A)$ denote the set of all multipliers on A.

(a) If $(L, R) \in M(A)$, then use the closed graph theorem to show that L and R must be bounded, and that $\|L\| = \|R\|$.

(b) Define operations and a norm on $M(A)$ so that $M(A)$ becomes a C^*-algebra which contains A as an ideal. (Use the example of A sitting in B as an ideal above for motivation.)

(c) An ideal A in B is called *essential* if the only ideal J in B such that $AJ = \{0\}$ is $J = \{0\}$. Show that A is an essential ideal in $M(A)$. Also show that if A is an essential ideal in a B, then there is an injective $*$-homomorphism of B into $M(A)$ which is the identity on A.

(d) Compute $M(A)$ for $A = C_0(X)$ and $A = \mathcal{K}(\mathcal{H})$.

3.3 Compact Operators

Definition 3.14. An operator T on a Hilbert space $\mathcal{H}$ is called *compact* if T takes the closed unit ball in $\mathcal{H}$ to a relatively compact subset of $\mathcal{H}$.

Notice that a compact operator is necessarily bounded. An equivalent formulation of compactness is that the image under T of any bounded sequence in $\mathcal{H}$ has a convergent subsequence.

We need a bit of "standard nonsense" concerning compact operators.

Lemma 3.15. *Suppose that $T \in B(\mathcal{H})$ is normal and compact.*

(a) *Every $\lambda \in \sigma(T) \setminus \{0\}$ is an eigenvalue.*

(b) *If $\lambda \neq 0$, then the eigenspace $E_\lambda = \{ h \in \mathcal{H} : Th = \lambda h \}$ is finite dimensional.*

(c) *$\sigma(T)$ is at most countable and has no nonzero accumulation points.*

Proof. Let $\lambda \in \sigma(T) \setminus \{0\}$. By Corollary 2.20 on page 35 there are unit vectors $\{h_n\}$ such that $(T - \lambda I)h_n \to 0$. Since T is compact, we may as well assume that $Th_n \to h$. But then we must have $\lambda h_n \to h$ and $\|h\| = |\lambda| \neq 0$.

But on the one hand, $T(\lambda h_n) = \lambda T(h_n) \to \lambda h$. On the other hand, $T(\lambda h_n) \to T(h)$. It follows that $Th = \lambda h$, and λ is an eigenvalue (since $h \neq 0$).

This proves part (a). Parts (b) and (c) follow from the observation that the image under T of any orthonormal family of eigenvectors with eigenvalues bounded away from zero can't have an accumulation point. The details are left as an exercise.[5] $\square$

One of the "biggies" in a first class on functional analysis is the Spectral Theorem for Normal Compact Operators. Here, we can derive it quickly via the Functional Calculus.

Theorem 3.16 (Spectral Theorem for Normal Compact Operators)**.** *Suppose that T is a compact normal operator on $\mathcal{H}$ with distinct nonzero eigenvalues $\{\lambda_i\}_{i\in I}$. Then*

$$T = \sum_{i\in I} \lambda_i P_i$$

where the P_i are pairwise orthogonal finite-rank projections in $C^*\big(\{T\}\big)$ and the convergence is in norm. In fact, P_i is the orthogonal projection onto the λ_i-eigenspace E_{λ_i}.

Proof. Let $\Psi : C\big(\sigma(T)\setminus\{0\}\big) \to C^*\big(\{T\}\big) \subset B(\mathcal{H})$ be the functional calculus isomorphism (see Corollary 2.19 on page 35). Since each λ_i is an isolated point in $\sigma(T)$ (by Lemma 3.15), the characteristic function f_i of $\{\lambda_i\}$ is in $C\big(\sigma(T)\setminus\{0\}\big)$ and

$$f = \sum_{i\in I} \lambda_i f_i$$

in $C\big(\sigma(T)\setminus\{0\}\big)$ (since $\lambda_i \to 0$ if $|I| = \infty$). Consequently,

$$T = \sum_{i\in I} \lambda_i \Psi(f_i)$$

in $B(\mathcal{H})$.

Clearly, each $\Psi(f_i)$ is a projection, and if $i \neq j$, then $\Psi(f_i)\Psi(f_j) = \Psi(f_i f_j) = 0$.

Let P_i be the projection onto the λ_i-eigenspace. Clearly, $\Psi(f_i) \leq P_i$.

Let $\mathcal{H}_0 := \ker T$ and $\mathcal{H}_i$ the space of $\Psi(f_i)$. It is not hard to see that $\overline{T(\mathcal{H})} = \bigoplus_{n=1}^{\infty} \mathcal{H}_i$. Moreover $T(\mathcal{H})^{\perp} = \ker(T^*)$. Since T is normal, $\ker(T^*) = \ker(T) = \mathcal{H}_0$ and $\mathcal{H}$ is the Hilbert space direct sum $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots$. Since $P_iP_j = 0$ if $i \neq j$, then it follows that $P_i(\mathcal{H}_k) = \{\,0\,\}$ if $i \neq k$. Hence $P_i = \Psi(f_i)$. $\square$

[5] Cross reference needed here.

Definition 3.17. A bounded operator T on $\mathcal{H}$ is called a *finite-rank* operator if it has finite-dimensional range $T(\mathcal{H})$. The collection of finite-rank operators on $\mathcal{H}$ is denoted by $B_f(\mathcal{H})$.

Example 3.18 (Rank-One Operators). If $e, f \in \mathcal{H}$, then we can define a rank-one operator by

$$\theta_{e,f}(h) := (h \mid f)e.$$

It is easy to check that $\theta_{e,f}$ is bounded and that $\|\theta_{e,f}\| = \|e\|\|f\|$.

If $T \in B_f(\mathcal{H})$, then let $\{e_1, \dots, e_n\}$ be an orthonormal basis for $T(\mathcal{H})$. Then

$$Th = \sum_{i=1}^{n} (Th \mid e_i)e_i = \sum_{i=1}^{n} (h \mid T^*e_i)e_i = \sum_{i=1}^{n} \theta_{e_i,T^*e_i}(h).$$

This is a good start towards proving the next result.

Lemma 3.19. *We have*

$$B_f(\mathcal{H}) = \operatorname{span}\{\, \theta_{h,k} : h, k \in \mathcal{H} \,\}.$$

In particular, $B_f(\mathcal{H})$ is a (not necessarily closed) two-sided self-adjoint ideal in $B(\mathcal{H})$.

Proof. Homework.[6] □

Remark 3.20. Recall that $h_i \to h$ weakly in $\mathcal{H}$ if and only if $(h_i \mid k) \to (h \mid k)$ for all $k \in \mathcal{H}$. That means $\theta_{e,f}(h_i) \to \theta_{e,f}(h)$ in norm. It follows that any $T \in B_f(\mathcal{H})$ is continuous from $\mathcal{H}$ with the weak topology to $\mathcal{H}$ with the norm topology. (In fact, the converse holds as well.[7])

Theorem 3.21 (Characterizations of Compactness)**.** *Let $\mathscr{B}_1$ be the* closed *unit ball in a Hilbert space $\mathcal{H}$. For $T \in B(\mathcal{H})$, the following are equivalent.*

(a) *T is the norm limit of finite-rank operators.*

(b) *$T|_{\mathscr{B}_1}$ is continuous from $\mathscr{B}_1$ with the weak topology to $\mathcal{H}$ with the norm topology.*

(c) *$T(\mathscr{B}_1)$ is compact.*

(d) *T is compact.*

[6] Cross reference needed here.
[7] Cross reference needed here.

Remark 3.22. In operator country, item (a) would be the natural definition of a compact operator — at least if there weren't historical precedent for the classical definition. An operator was called *completely continuous* if it takes weakly convergent sequences to norm convergent sequences. Since a weakly convergent sequence is bounded by the Principle of Uniform Boundedness, it follows, at least in the case where $\mathcal{H}$ is separable and hence $\mathscr{B}_1$ is metrizable, item (b) just says that T is completely continuous.

For the proof, we are going to need a technical result. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Let Λ be the collection of finite subsets of I directed by inclusion. Recall that if $h \in \mathcal{H}$, then

$$h = \sum_{i\in I} (h \mid e_i)e_i = \lim_{\lambda\in\Lambda} \sum_{i\in\lambda} (h \mid e_i)e_i. \tag{3.2}$$

Lemma 3.23. *Let $\{e_i\}_{i\in I}$ and Λ be as above. Let P_λ be the projection onto the finite-dimensional subspace* $\operatorname{span}\{\, e_i : i \in \lambda \,\}$*. Then for all $h \in \mathcal{H}$, we have* $\lim_\lambda P_\lambda h = h$.

Proof. This follows almost immediately from (3.2). The details are left as an exercise. □

Proof of Theorem 3.21. (a)$\Longrightarrow$(b): Suppose that $h_i \to h$ weakly in $\mathscr{B}_1$. Fix $\epsilon > 0$, and let S be a finite-rank operator such that $\|T - S\| < \epsilon/3$. Since $\{h_i\} \subset \mathscr{B}_1$, we have

$$\|Th_i - Th\| \le 2\|T - S\| + \|Sh_i - Sh\|.$$

Since S is weak-norm continuous, there is an i_0 such that $i \ge i_0$ implies $\|Sh_i - Sh\| < \epsilon/3$. The result follows.

(b)$\Longrightarrow$(c): Follows since $\mathscr{B}_1$ is weakly compact.

(c)$\Longrightarrow$(d): Immediate.

(d)$\Longrightarrow$(a): We are going to invoke Lemma 3.23. Since each P_λ, and therefore $P_\lambda T$, is finite-rank, it will suffice to prove that $P_\lambda T \to T$ if T is compact. Suppose to the contrary, that $P_\lambda T \not\to T$. Then there is an $\epsilon > 0$ such that $\|P_\lambda T - T\|$ is frequently greater than ϵ. Therefore we can pass to a subnet, relabel, and assume that there are unit vectors h_λ such that

$$\|P_\lambda T(h_\lambda) - T(h_\lambda)\| \ge \epsilon \quad \text{for all } \lambda. \tag{3.3}$$

Since T is compact, there is no harm in also assuming that $T(h_\lambda) \to h$. But then

$$\begin{aligned} \|P_\lambda T(h_\lambda) - T(h_\lambda)\| &= \|(P_\lambda - I)T(h_\lambda)\| \\ &= \|(P_\lambda - I)(T(h_\lambda) - h) + (P_\lambda - I)h\| \\ &\le \|T(h_\lambda) - h\| + \|P_\lambda h - h\|. \end{aligned}$$

This eventually contradicts (3.3). □

Remark 3.24. The complicated part of the proof of Theorem 3.21 — namely that a compact operator is the limit of finite-rank operators follows immediately if the operator is normal by the Spectral Theorem for Normal Compact operators. The general case would follow immediately if I knew that the adjoint of a compact operator was compact. (Of course this follows once we have proved Theorem 3.21. Unfortunately, I don't know a proof that T compact implies T^* compact that is any easier than the above.)

Corollary 3.25. *The collection $\mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$ is a (norm closed two-sided) ideal in $B(\mathcal{H})$. In particular, $\mathcal{K}(\mathcal{H})$ is a C^*-algebra.*

Proof. We know that $B_f(\mathcal{H})$ is a two-sided ideal in $B(\mathcal{H})$ by Lemma 3.19, and $\mathcal{K}(\mathcal{H}) = \overline{B_f(\mathcal{H})}$ by the theorem. □

Exercises

E 3.3.1. Finish the proof of parts (b) and (c) of Lemma 3.15 on page 56.

E 3.3.2. Prove Lemma 3.19 on page 58.

E 3.3.3. Show that a linear operator $T : \mathcal{H} \to \mathcal{H}$ which is continuous from $\mathcal{H}$ with the weak topology to $\mathcal{H}$ with the norm topology must be a finite-operator.

—Answer on page 139

E 3.3.4. Finish the proof of Lemma 3.23 on the preceding page.

3.4 Representations of the Compacts

In this section, $\mathscr{A}$ will usually be a C^*-subalgebra of $\mathcal{K}(\mathcal{H})$. C^*-algebras that sit inside some $B(\mathcal{H})$ are sometimes called *Concrete C^*-algebras*.

Definition 3.26. If A is a C^*-algebra, then $p \in A$ is called a *projection* if $p = p^2 = p^*$. We say that a projection $q \in A$ is a *subprojection* of p if $pq = qp = q$. We say that p is a minimal projection in A if p has no nontrivial subprojections in A.

Lemma 3.27. *Let $\mathscr{A} \subset \mathcal{K}(\mathcal{H})$ be a C^*-subalgebra, and let $p \in \mathscr{A}$ be a projection. Then p is minimal in $\mathscr{A}$ if and only if*

$$p\mathscr{A}p = \{\, \lambda p : \lambda \in \mathbf{C} \,\}.$$

Furthermore, every projection in $\mathscr{A}$ is finite-rank and the orthogonal sum of minimal projections.

Proof. If $p\mathscr{A}p = \mathbf{C}p$, then p is certainly minimal.

On the other hand, if p is minimal and if $T \in \mathscr{A}_{s.a.}$, then the Spectral Theorem implies that

$$pTp = \sum_i \lambda_i P_i$$

where the spectral projections P_i belong to $\mathscr{A}$. Clearly we must have each $P_i \leq p$. Hence $pTp = \lambda p$. This proves the first statement.

The result follows from the observation than a compact projection must be finite-rank and induction. □

Theorem 3.28. *Suppose that $\mathscr{A}$ is an* irreducible subalgebra *of $\mathcal{K}(\mathcal{H})$. (That is, the identity representation* $\mathrm{id} : \mathscr{A} \to B(\mathcal{H})$ *is irreducible.) Then $\mathscr{A} = \mathcal{K}(\mathcal{H})$.*

Proof. If $T \in \mathscr{A}_{s.a.}$, then $T = \sum_i \lambda_i P_i$ and each P_i belongs to $\mathscr{A}$. Therefore $\mathscr{A}$ contains projections and therefore a minimal projection p.

I claim that $\dim p = 1$. Let $h, k \in p\mathcal{H}$ with $h \perp k$ and $\|h\| = 1$. Suppose $T \in \mathscr{A}$. Then $pTp = \lambda p$ and

$$(k \mid Th) = (k \mid pTph) = \bar{\lambda}(k|h) = 0.$$

Therefore $k \in [\mathscr{A}h]^\perp$. But $\mathscr{A}$ irreducible implies that $[\mathscr{A}h] = \mathcal{H}$, so that $k = 0$. This proves the claim.

Note that $p(f) = \theta_{h,h}(f) = (f \mid h)h$.

Let q be any rank-one projection in $B(\mathcal{H})$. Say, $q(f) = \theta_{k,k}(f) = (f \mid k)k$ with $\|k\| = 1$. Since $[\mathscr{A}h] = \mathcal{H}$, there are $T_n \in \mathscr{A}$ such that $T_n h \to k$. We can assume that $\|T_n h\| = 1$. Then $T_n p T_n^* \in \mathscr{A}$ and

$$T_n p T_n^*(f) = T_n\big((T_n^* f \mid h)h\big) = (T_n^* f \mid h)T_n h = (f \mid T_n h)T_n h = \theta_{T_n h, T_n h}(f).$$

In sum, $T_n p T_n^*$ is a projection and for any $f \in \mathcal{H}$,

$$\begin{aligned} |T_n p T_n^*(f) - q(f)| &= |(f \mid T_n h)T_n h - (f \mid k)k)| \\ &= |(f \mid T_n h - k)T_n h + (f \mid k)(T_n h - k)| \\ &\leq \|f\|\|T_n h - k\|\|T_n h\| + \|f\|\|k\|\|T_n h - k\| \\ &\leq 2\|f\|\|T_n h - k\|. \end{aligned}$$

It follows that $\|T_n p T_n^* - q\| \leq 2\|T_n h - k\|$. Therefore $T_n p T_n^* \to q$ and $q \in \mathscr{A}$.

Since $\mathscr{A}$ contains every rank-one projection, it contains every finite-rank projection. By the Spectral Theorem for Compact Normal Operators, $\mathscr{A}$ must contain every self-adjoint compact operator. Hence $\mathscr{A} = \mathcal{K}(\mathcal{H})$. □

Definition 3.29. A C^*-algebra is called *simple* if it has no nontrivial (closed) ideals.

It is worth pointing out that a simple C^*-algebra can have nontrivial ideals that are not closed. For example, although our next result shows that the compacts are always simple, $\mathcal{K}(\mathcal{H})$ has lots of nonclosed proper ideals if $\dim \mathcal{H} = \infty$ — the finite-rank operators are an example.

Corollary 3.30. *$\mathcal{K}(\mathcal{H})$ is simple.*

Proof. Suppose that $J \subset \mathcal{K}(\mathcal{H})$ is a nonzero ideal. Since $\mathrm{id} : \mathcal{K}(\mathcal{H}) \to B(\mathcal{H})$ is irreducible, $\mathrm{id}|_J$ is too (Theorem 3.12 on page 54). But then $J = \mathcal{K}(\mathcal{H})$ by Theorem 3.28. □

Proposition 3.31. *Suppose that $\mathscr{A}$ is a C^*-subalgebra of $\mathcal{K}(\mathcal{H})$. Let p be a minimal projection in $\mathscr{A}$ and let h be a unit vector in $p\mathcal{H}$. If $\mathcal{H}_0 := [\mathscr{A}h]$, then $\mathscr{A}|_{\mathcal{H}_0}$ is irreducible and $\mathscr{A}|_{\mathcal{H}_0} \cong \mathcal{K}(\mathcal{H}_0)$.*

Proof. Since $\mathcal{H}_0$ is invariant, each $T \in \mathscr{A}$ defines an operator on $\mathcal{H}_0$ which is clearly compact. Furthermore $T \mapsto T|_{\mathcal{H}_0}$ is a $*$-homomorphism of $\mathscr{A}$ into $\mathcal{K}(\mathcal{H}_0)$ with range $\mathscr{A}|_{\mathcal{H}_0}$. Thus by Theorem 3.28, it will suffice to see that $\mathscr{A}|_{\mathcal{H}_0}$ is irreducible.

Let

$$(\mathscr{A}|_{\mathcal{H}_0})' = \{\, T \in B(\mathcal{H}_0) : TS|_{\mathcal{H}_0} = S|_{\mathcal{H}_0}T \text{ for all } S \in \mathscr{A}\,\}.$$

By Proposition 3.6, it suffices to show that each $R \in (\mathscr{A}|_{\mathcal{H}_0})'$ is a scalar operator. Replacing R by $R-(Rh \mid h)I$ means that we can assume $(Rh \mid h) = 0$. It will suffice to show that such an R is the zero operator.

Since p is minimal in $\mathscr{A}$, we know that $pT^*Sp \in \mathbf{C}p$ for all $T, S \in \mathscr{A}$. But then

$$\begin{aligned}(RSh \mid Th) &= (RSph \mid Tph)\\ &= (pT^*RSph \mid h)\\ &= (RpT^*Sph \mid h)\\ &= \lambda(Rh \mid h)\\ &= 0.\end{aligned}$$

Since $\mathscr{A}h$ is dense in $\mathcal{H}_0$, it follows that $R = 0$. $\square$

Definition 3.32. If $\pi_i : A \to B(\mathcal{H}_i)$ is a representation, then $\pi_1 \oplus \cdots \oplus \pi_n$ is the representation of A on $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ given by

$$(\pi_1 \oplus \cdots \oplus \pi_n)(a)(h_1, \dots, h_n) = \big(\pi_1(a)h_1, \cdots, \pi_n(a)(h_n)\big).$$

We call $\pi_1 \oplus \cdots \oplus \pi_n$ the *direct sum* of the π_i. If $n \in \mathbf{N}$, then $n \cdot \pi := \bigoplus_{i=1}^n \pi$ on $\bigoplus_{i=1}^n \mathcal{H}_i$.

More generally, if $\mathcal{H}_i$ is a Hilbert space for each $i \in I$, then we define the Hilbert space direct sum by

$$\bigoplus_{i\in I} \mathcal{H}_i = \{\, (h_i) \in \prod \mathcal{H}_i : \sum_{i\in I} \|h_i\|^2 < \infty \,\}.$$

(The inner product on $\bigoplus_{i\in I} \mathcal{H}_i$ is given by $(h \mid k) = \sum_{i\in I}(h_i \mid k_i)$.) Then we can define $\bigoplus_{i\in I} \pi_i$ and $|I| \cdot \pi$ in analogy with Definition 3.32.

If V is a *closed* invariant subspace for $\pi : A \to B(\mathcal{H})$, then we get a representation π_V of A on $B(V)$ via

$$\pi_V(a) = \pi(a)|_V.$$

We call π_V a *subrepresentation* of π and write $\pi_V \le \pi$. Note that $V^\perp$ is also invariant and that (after identifying $\mathcal{H}$ with $V \oplus V^\perp$)

$$\pi = \pi_V \oplus \pi_{V^\perp}.$$

We also write $\pi_{V^\perp} = \pi \ominus \pi_V$.

Theorem 3.33. *Suppose that $\mathscr{A}$ is a C^*-subalgebra of $\mathcal{K}(\mathcal{H}_1)$ and that π is a nondegenerate representation of $\mathscr{A}$ in $B(\mathcal{H}_2)$. Then there is a pairwise orthogonal family $\{\pi_i\}_{i\in I}$ of irreducible subrepresentations of π such that*

$$\pi = \bigoplus_{i\in I} \pi_i.$$

Moreover, each π_i is equivalent to a subrepresentation of $\mathrm{id} : \mathscr{A} \to B(\mathcal{H}_1)$.

Proof. Let $T \in \mathscr{A}_{s.a.}$ be such that $\pi(T) \neq 0$. Then $\pi(q) \neq 0$ for some spectral projection associated to T. It follows that $\mathscr{A}$ contains a minimal projection p such that $\pi(p) \neq 0$. Since p is minimal, we can define $f : \mathscr{A} \to \mathbf{C}$ by

$$f(T)p = pTp.$$

Let h_1 and h_2 be unit vectors such that $h_1 \in p\mathcal{H}_1$ and $h_2 \in \pi(p)\mathcal{H}_2$. Let

$$V := [\pi(\mathscr{A})h_2].$$

By Proposition 3.31, the subrepresentation $\mathrm{id}_{[\mathscr{A}h_1]}$ is irreducible. I claim $\pi_V \sim \mathrm{id}_{[\mathscr{A}h_1]}$.

To see this, compute that

$$\begin{aligned} \|\pi(T)h_2\|^2 &= \|\pi(Tp)h_2\|^2 \\ &= \big(\pi(pT^*Tp)h_2 \mid h_2\big) \\ &= f(T^*T)\big(\pi(p)h_2 \mid h_2\big) \\ &= f(T^*T) \\ &= \big(pT^*Tph_1 \mid h_1\big) \\ &= \|Th_1\|^2. \end{aligned}$$

Therefore $Th_1 \mapsto \pi(T)h_2$ extends to a unitary operator $U : [\mathscr{A}h_1] \to [\pi(\mathscr{A})h_2]$. If $S \in \mathscr{A}|_{[\mathscr{A}h_1]}$, then

$$USTh_1 = \pi(ST)h_2 = \pi_V(S)\pi(T)h_2 = \pi_V(S)U(Th_1).$$

This shows that

$$US = \pi_V(S)U,$$

and we have shown that $\pi_V \sim \mathrm{id}_{[\mathscr{A}h_1]}$

We have shown that every nondegenerate representation of $\mathscr{A}$ contains a subrepresentation equivalent to an irreducible subrepresentation of $\mathrm{id} : \mathscr{A} \to B(\mathcal{H}_1)$. Apply this to $\pi \ominus \pi_V$. Continuing in this way — using Zorn's Lemma — we produce a maximal family $\{\pi_i\}_{i\in I}$ of orthogonal subrepresentations of π each equivalent to an irreducible subrepresentation of id. By the maximality, $\pi = \bigoplus_{i\in I} \pi_i$. □

Corollary 3.34. *Every non-degenerate representation of $\mathcal{K}(\mathcal{H})$ is equivalent to a multiple of the identity representation* $\mathrm{id} : \mathcal{K}(\mathcal{H}) \to B(\mathcal{H})$.

Proof. Since id is irreducible, it has only itself as a (nonzero) subrepresentation. Thus if $\pi : \mathcal{K}(\mathcal{H}) \to B(\mathcal{H}_\pi)$ is nondegenerate, then Theorem 3.33 implies that $\pi = \bigoplus_{i\in I} \pi_i$ with each $\pi_i \sim \mathrm{id}$. Thus there are unitaries $U : \mathcal{H} \to \mathcal{H}_{\pi_i} \subset \mathcal{H}_\pi$ such that

$$T = U_i^* \pi_i(T) U_i \quad \text{for all } T \in \mathcal{K}(\mathcal{H}).$$

Then if $n := |I|$, we have

$$n \cdot \mathrm{id}(T) = U^* \pi(T) U,$$

where $U = \bigoplus_{i\in I} U_i : \bigoplus_{i\in I} \mathcal{H} \to \mathcal{H}_\pi$. □

Corollary 3.35. *Every irreducible representation of $\mathcal{K}(\mathcal{H})$ is equivalent to the identity representation.*

Corollary 3.36. *Suppose that $\varphi : \mathcal{K}(\mathcal{H}_1) \to \mathcal{K}(\mathcal{H}_2)$ is a surjective $*$-homomorphism. Then there is a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\varphi(T) = UTU^*$ for all $T \in \mathcal{K}(\mathcal{H}_1)$. If $\psi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is a $*$-isomorphism, then we also have $\psi(T) = UTU^*$ for a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$.*

Proof. The first assertion follows immediately once we realize that φ is an irreducible representation of $\mathcal{K}(\mathcal{H}_1)$ on $\mathcal{H}_2$.

For the second, ψ is still and irreducible representation and $\psi|_{\mathcal{K}(\mathcal{H}_1)}$ is nonzero. Hence $\psi|_{\mathcal{K}(\mathcal{H}_1)}$ is irreducible. Thus, as above, there is a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\psi(T) = UTU^*$ for all $T \in \mathcal{K}(\mathcal{H}_1)$. But $\tilde{\psi}(T) = UTU^*$ defines an irreducible representation of $B(\mathcal{H}_1)$ extending $\psi|_{\mathcal{K}(\mathcal{H}_1)}$. Hence ψ and $\tilde{\psi}$ are equal by the uniqueness assertion in Theorem 3.12 on page 54. □

Exercises

E 3.4.1. Let $\pi : A \to \mathscr{B}(\mathcal{H})$ be a (possibly degenerate) representation. Let $\mathcal{H}_0 := \overline{\operatorname{span}}\{\, \pi(a)h : a \in A \text{ and } h \in \mathcal{H} \,\}$. We call $\mathcal{H}_0$ the essential subspace of π. Show that $\mathcal{H}_0$ is invariant and let $\operatorname{ess}\pi$ be the corresponding subrepresentation of π. Show that $\pi = \operatorname{ess}\pi \oplus 0$, where "0" denotes the zero representation on $\mathcal{H}_0^\perp$.

3.5 Classes of C^*-Algebras

Definition 3.37. A C^*-algebra A is called a CCR algebra if the image of every irreducible representation of A consists of compact operators.

Remark 3.38. It follows from Theorem 3.28 on page 61 that if A is CCR, then $\pi(A) = \mathcal{K}(\mathcal{H}_\pi)$ for every irreducible representation of A.

Remark 3.39 (Other Terminology). The use of "CCR" is not universal. I am told that it stands for "completely continuous representation", but in fact, to me, it is just a sequence of three letters that stands for a particularly well-behaved representation theory. In his seminal book [Dix77], Dixmier used the term *liminaire* which is often translated as *liminal*. Gert Pedersen suggested that since "liminarie" is not unrelated to "preliminarie", a better translation might be *liminary* (as in "preliminary").

Example 3.40. Of course $\mathcal{K}(\mathcal{H})$ and $C_0(X)$ are both examples of CCR algebras. More generally, $C_0(X, \mathcal{K}(\mathcal{H}))$ is CCR[8] as is any C^*-algebra with all finite dimensional irreducible representations.

It will be convenient in the sequel to introduce a bit of notation. I'll use $\operatorname{Irr}(A)$ to denote the collection of all irreducible representations of A and $\operatorname{Rep}(A)$ to denote the collection of all *nondegenerate* representations of A.

Proposition 3.41. *Suppose that A is a CCR algebra. Then the kernel of every irreducible representation is a maximal ideal in A. If π and ρ are irreducible representations of A such that $\ker \pi \subset \ker \rho$, then $\pi \sim \rho$.*

Proof. If $\pi \in \operatorname{Irr}(A)$, then π induces a $*$-isomorphism of $A/\ker \pi$ onto $\mathcal{K}(\mathcal{H}_\pi)$. Since $\mathcal{K}(\mathcal{H}_\pi)$ is simple, $\ker \pi$ must be a maximal ideal.

If in addition, $\rho \in \operatorname{Irr}(A)$ and $\ker \pi \subset \ker \rho$, then we can define a $*$-homomorphism $\sigma : \mathcal{K}(\mathcal{H}_\pi) \to \mathcal{K}(\mathcal{H}_\rho)$ by $\sigma\bigl(\pi(a)\bigr) = \rho(a)$. Then σ is an irreducible representation of $\mathcal{K}(\mathcal{H}_\pi)$ and σ is equivalent to $\mathrm{id} : \mathcal{K}(\mathcal{H}_\pi) \to B(\mathcal{H}_\pi)$. Thus, there is a unitary $U : \mathcal{H}_\pi \to \mathcal{H}_\rho$ such that

$$\pi(a) = U^*\sigma\bigl(\pi(a)\bigr)U = U^*\rho(a)U \quad \text{for all } a \in A. \qquad \square$$

Example 3.42. Let $\mathcal{H} = \ell^2$ with its usual orthonormal basis $\{e_n\}_{n=1}^\infty$. The *unilateral shift* on $\mathcal{H}$ is the operator S which sends e_n to e_{n+1}. Note the S is a non-unitary isometry (so that $S^*S = I \neq SS^*$). Let $C^*(S)$ be the unital C^*-algebra generated

[8] This is not so obvious at this point. Perhaps add an exercise or see [RW98, Example A.24].

by S. It is an exercise[9] to see that $C^*(S)$ contains $\mathcal{K}(\ell^2)$, and that the quotient $C^*(S)/\mathcal{K}(\ell^2)$ is isomorphic to $C(\mathbf{T})$. In particular, the identity representation is irreducible and contains noncompact operators. Thus $C^*(S)$ is not CCR.

Lemma 3.43. *Suppose that A is a C^*-algebra. Then*

$$\mathrm{CCR}(A) := \bigcap_{\pi\in\mathrm{Irr}(A)} \{\, a \in A : \pi(a) \in \mathcal{K}(\mathcal{H}_\pi) \,\}$$

is a CCR ideal in A. If J is any other CCR ideal, then $J \subset \mathrm{CCR}(A)$.

Proof. For each $\pi \in \mathrm{Irr}(A)$, $K_\pi := \{\, a \in A : \pi(a) \in \mathcal{K}(\mathcal{H}_\pi) \,\} = \pi^{-1}\big(\mathcal{K}(\mathcal{H}_\pi)\big)$ is an ideal in A. Since $\mathrm{CCR}(A) = \bigcap_{\pi\in\mathrm{Irr}(A)} K_\pi$, $\mathrm{CCR}(A)$ is an ideal.

Let ρ be an irreducible representation of $\mathrm{CCR}(A)$. Then ρ is the restriction to $\mathrm{CCR}(A)$ of some $\pi \in \mathrm{Irr}(A)$. Thus $\rho(\mathrm{CCR}(A)) = \pi(\mathrm{CCR}(A)) \subset \mathcal{K}(\mathcal{H}_\rho)$, and the last containment is an equality as ρ is irreducible. It follows that $\mathrm{CCR}(A)$ is a CCR algebra.

The last assertion is left as an exercise.[10] $\square$

Definition 3.44. A C^*-algebra A is called a GCR algebra if $\mathrm{CCR}(A/J) \neq \{0\}$ for all proper ideals J of A.

Remark 3.45. I am not quite sure what “GCR” stands for, but have always assumed that the ‘G’ stood for generalized. Dixmier used *postliminaire* which has been translated as *postliminal* and *postliminary*.

Example 3.46. Suppose that J is a nonzero proper ideal in the C^*-algebra $C^*(S)$ generated by the unilateral shift (see Example 3.42). Then either $J = \mathcal{K}(\ell^2)$ or J properly contains $\mathcal{K}(\ell^2)$. In the latter case, $C^*(S)/J$ is commutative. Hence, $C^*(S)$ is a GCR algebra.

Proposition 3.47. *Suppose that A is a GCR algebra. Then*

$$\pi(A) \supset \mathcal{K}(\mathcal{H}_\pi) \quad \textit{for all } \pi \in \mathrm{Irr}(A). \tag{3.4}$$

Furthermore, if ρ is another irreducible representation such that $\ker\pi = \ker\rho$, *then* $\pi \sim \rho$.

[9] Cross reference needed here.
[10] Cross reference needed here.

Remark 3.48. It is an exercise to show that A is GCR if and only if every irreducible representation has at least one nonzero compact operator in its range.[11] However, it is a very deep result in the theory that (3.4) actually characterizes GCR algebras. In fact, GCR algebras are also characterized by the second assertion in Proposition 3.47. These results are beyond the scope of these notes. For the details and/or further references, see [Ped79, Theorem 6.8.7] or [Dix77, Theorem 9.1].

Proof of Proposition 3.47. Since $\pi(A)$ is isomorphic (as a C^*-algebra) to $A/\ker\pi$, it follows that $J := \mathrm{CCR}\big(\pi(A)\big) \neq \{0\}$. But $\mathrm{id} : \pi(A) \to B(\mathcal{H}_\pi)$ is irreducible and $\mathrm{id}\,|_{J\cap\mathcal{K}(\mathcal{H}_\pi)} \neq 0$. Therefore $\mathrm{id}_{J\cap\mathcal{K}(\mathcal{H}_\pi)}$ is irreducible and $J \cap \mathcal{K}(\mathcal{H}_\pi) = \mathcal{K}(\mathcal{H}_\pi)$ by Theorem 3.12 on page 54 and Theorem 3.28 on page 61. This proves the first assertion.

Now suppose $\pi, \rho \in \mathrm{Irr}(A)$ and that $\ker\pi = \ker\rho$. Define $\sigma : \pi(A \to B(\mathcal{H}_\rho)$ by $\sigma\big(\pi(a)\big) = \rho(a)$.

Since σ is irreducible and $\ker\sigma = \{0\}$ and since $\mathcal{K}(\mathcal{H}_\pi) \subset \pi(A)$, $\sigma|_{\mathcal{K}(\mathcal{H}_\pi)}$ is irreducible. Therefore $\sigma|_{\mathcal{K}(\mathcal{H}_\pi)}$ is equivalent to $\mathrm{id} : \mathcal{K}(\mathcal{H}_\pi) \to B(\mathcal{H}_\pi)$. By Theorem 3.12, we must have σ equivalent to $\mathrm{id} : \pi(A) \to B(\mathcal{H}_\pi)$. Untangling, this means that $\pi \sim \rho$. $\square$

For the purposes of these notes, a set $I = \{\,\alpha : 0 \le \alpha \le \alpha_0\,\}$ of *ordinals* is a totally ordered set in which each element has a smallest element. Therefore each $\alpha < \alpha_0$ has an *immediate successor*: $\alpha + 1 := \min\{\,\gamma : \gamma > \alpha\,\}$. If $\alpha = \beta + 1$ for some β, then β is called an *immediate predecessor* for α. If α does not have an immediate predecessor, it is called a *limit ordinal*. Note that any set I of ordinals has a smallest element, usually written 0. In set theory land, its successors are usually written 1, 2, ... If I is infinite, there is a first infinite ordinal ω, which is a limit ordinal. After that, the structure of ordinals gets very "interesting". However, this is purely notation. For example, the set

$$I = \{\,1 - \frac{1}{n} : n \in \mathbf{N}\,\} \cup \{\,2 - \frac{1}{n} : n \in \mathbf{N}\,\} \cup \{2\} \tag{3.5}$$

Is a set of ordinals (in the order inherited from $\mathbf{R}$) with limit ordinals 1 and 2.

Definition 3.49. A *composition series* in a C^*-algebra A is a family of ideals $\{J_\alpha\}$ indexed by ordinals $I = \{0 \le \alpha \le \alpha_0\}$ such that

(a) $J_\alpha \subsetneq J_{\alpha+1}$ for all $\alpha < \alpha_0$.
(b) $J_0 = \{0\}$ and $J_{\alpha_0} = A$.

[11] Cross reference needed here.

(c) If β is a limit ordinal, then $J_\beta = \overline{\bigcup_{\alpha<\beta} J_\alpha}$.

Example 3.50. Let $A = C\big([0,2]\big)$ and for each i in the set I from (3.5), let J_i be the ideal in A given by

$$J_i = \{\, f \in A : f(x) = 0 \text{ if } x > i \,\}$$

(This means that J_i is the ideal of functions vanishing on $[i,2]$ if $i < 2$ and $J_2 = A$.) Then $\{J_i\}_{i\in I}$ is a composition series for $A_0 = \{\, f \in C\big([0,2]\big) : f(2) = 0 \,\}$.

Theorem 3.51. *Every GCR algebra has a unique composition series $\{\, J_\alpha : 0 \le \alpha \le \alpha_0 \,\}$ such that $J_{\alpha+1}/J_\alpha = \mathrm{CCR}(A/J_\alpha)$ for $\alpha < \alpha_0$. Conversely, if A has a composition series $\{\, I_\gamma : 0 \le \gamma \le \gamma_0 \,\}$ such that $I_{\gamma+1}/I_\gamma$ is CCR, then A is GCR.*

While we technically need ordinals for the next results, it should be kept in mind that in practice, our sets of ordinals are going be finite. For example, consider the C^*-algebra $C^*(S)$ generated by the unilateral shift (as in Examples 3.42 and 3.46). It has $\{0, \mathcal{K}(\ell^2), C^*(S)\}$ as a composition series (of length just 2). Since $C^*(S)/\mathcal{K}(\ell^2)$ is abelian — and therefore CCR — Theorem 3.51 implies $C^*(S)$ is GCR.

Proof of Theorem 3.51. Taken from [Arv76][12] □

Exercises

E 3.5.1. Let $\pi : A \to B(\mathcal{H})$ be an irreducible representation of a C^*-algebra A. Suppose that $\pi(A) \cap \mathcal{K}(\mathcal{H}) \neq \{\,0\,\}$. Show that $\pi(A) \supset \mathcal{K}(\mathcal{H})$. ("If the range of an irreducible representation contains one nonzero compact operator, then it contains them all.") If you want a hint, look over the proof of Proposition 3.47 on page 67.

—Answer on page 139

E 3.5.2. Complete the proof of Lemma 3.43 on page 67. That is, show that $\mathrm{CCR}(A)$ is the largest CCR ideal in A in the sense that if J is any CCR ideal in A, then $J \subset \mathrm{CCR}(A)$.

E 3.5.3. Let $\{\, J_\alpha : 0 \le \alpha \le \alpha_0 \,\}$ be a composition series for a *separable* C^*-algebra A. Show that α_0 is countable. (Recall that α_0 is called countable if $\{\, \alpha : 0 \le \alpha < \alpha_0 \,\}$ is countable. Also, for each $\alpha < \alpha_0$ notice that you can find $a_\alpha \in J_{\alpha+1}$ such that $\|a_{\alpha+1} - a\| \ge 1$ for all $a \in J_\alpha$.)

[12] Do it.

—Answer on page 139

E 3.5.4. Suppose that $\{ J_\alpha : 0 \leq \alpha \leq \alpha_0 \}$ is a composition series for a C^*-algebra A. A nondegenerate representation π if A is said to *live on the subquotient* $J_{\alpha+1}/J_\alpha$ if π is the canonical extension to A of a representation π' of $J_{\alpha+1}$ such that $\ker \pi' \supset J_\alpha$. That is, π' must be of the form $\pi' = \rho \circ q_\alpha$ where $q_\alpha : J_{\alpha+1} \to J_{\alpha+1}/J_\alpha$ is the natural map, and ρ is a nondegenerate representation of $J_{\alpha+1}/J_\alpha$. Show that every *irreducible* representation of A lives on a subquotient so that the spectrum of A can be identified with the disjoint union of the spectra of the subquotients $J_{\alpha+1}/J_\alpha$ for $\alpha < \alpha_0$.

E 3.5.5. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{ e_n \}_{n=1}^\infty$. Let S be the *unilateral shift* operator $S \in B(\mathcal{H})$ defined by $S(e_n) = e_{n+1}$ for all n. Finally, let A be the unital C^*-algebra generated by S (i.e., $A = C^*(S)$), and let $\mathbf{T} = \{ z \in \mathbf{C} : |z| = 1 \}$.

(a) Show that $S^*S - SS^* = P$, where P is the rank-one projection onto $\mathbf{C}e_1$.

(b) Show that A is irreducible, and that $\mathcal{K}(\mathcal{H}) \subseteq A$.

(c) Show that if $\alpha \in \mathbf{T}$, then there is a unitary operator U in $B(\mathcal{H})$ such that $USU^* = \alpha S$.

(d) Show that the quotient $A/\mathcal{K}(\mathcal{H})$ is $*$-isomorphic to $C(\mathbf{T})$.

(e) Conclude that A is GCR, but not CCR.

(f) Describe the (equivalence classes) of irreducible representations of A.

Suggestions: In part (b), show that e_1 is cyclic for the identity representation of A. Now observe that if V is a closed invariant subspace for A, then either $e_1 \in V$ or $e_1 \in V^\perp$. In part (d), notice that the image of S in the quotient is unitary (hence normal), generates, and has spectrum $\mathbf{T}$.

—Answer on page 139

E 3.5.6. Let $\mathcal{H}$ be a *separable* infinite dimensional Hilbert space. Recall that $T \in B(\mathcal{H})$ is said to be *bounded below* if there is an $\epsilon > 0$ such that $|T\xi| \geq \epsilon |\xi|$ for all $\xi \in \mathcal{H}$.

(a) Show that if $T \in B(\mathcal{H})_{\text{s.a.}}$ is bounded from below, then T has a bounded inverse.

(b) If $T \in B(\mathcal{H})_{\text{s.a.}}$ and $\epsilon > 0$, then define

$$M_\epsilon = \overline{\operatorname{span}}\{\, f(T)\xi : \xi \in \mathcal{H},\ f \in C\big(\sigma(T)\big), \text{ and } f(\lambda) = 0 \text{ if } |\lambda| \leq \epsilon \,\}.$$

Show that $|T\xi| \geq \epsilon|\xi|$ for all $\xi \in M_\epsilon$, and that $TM_\epsilon = M_\epsilon$.

(c) Show that if $T \in B(\mathcal{H})_{\text{s.a.}}$ is not compact, then there is an $\epsilon > 0$ so that M_ϵ is infinite dimensional. In particular, conclude that there is a partial isometry $V \in B(\mathcal{H})$ such that V^*TV has a bounded inverse.

(d) Show that $\mathcal{K}(\mathcal{H})$ is the only non-zero proper closed ideal in $B(\mathcal{H})$.

(e) Assuming that any C^*-algebra has irreducible representations, conclude that $B(\mathcal{H})$ is not a GCR algebra.

—Answer on page 139

Chapter 4

The Gelfand-Naimark Theorem

Finding irreducible representations of a given C^*-algebra can be very challenging. We've only looked at some very basic situations so far.

Example 4.1 (Subalgebra of the Compacts). If $\mathscr{A}$ is a subalgebra of $\mathcal{K}(\mathcal{H})$, then every irreducible representation occurs, up to equivalence of course, as a subrepresentation of the identity representation $\mathrm{id} : \mathscr{A} \to B(\mathcal{H})$.

Example 4.2 (The Unilateral Shift). For $C^*(S)$ sitting in $B(\ell^2)$, we note that $C^*(S)$ has one-dimensional irreducible representations that can't be subrepresentations of the irreducible representation $\mathrm{id} : C^*(S) \to B(\ell^2)$.

Example 4.3. If M is the representation of $C_0(\mathbf{R})$ on $L^2(\mathbf{R})$ given my pointwise multiplication, then M has no irreducible subrepresentations whatsoever.[1]

4.1 The GNS Construction

Definition 4.4. A linear functional $f : A \to \mathbf{C}$ is called *positive* if $f(a^*a) \geq 0$ for all $a \in A$.

Example 4.5. Suppose that π is a representation of a C^*-algebra A on $\mathcal{H}$. Then if $h \in \mathcal{H}$,

$$f(a) := (\pi(a)h \mid h)$$

is a positive linear functional on A.

[1]Cross reference needed here.

Recall that a *sesquilinear form* on a (complex) vector space V is a function $q : V \times V \to \mathbf{C}$ such that is linear in its first variable and conjugate linear in the second. Note that a sesquilinear form satisfies the polarization identity

$$q(v,w) = \frac{1}{4}\sum_{n=0}^{3} i^n q(v + i^n w, v + i^n w). \tag{4.1}$$

(The polarization identity of Remark 2.21 is a special case: consider $q(h,k) := \big(T(h) \mid k\big)$.) We say that q is *self-adjoint* if $q(v,w) = \overline{q(w,v)}$ for all $v, w \in V$. We say that q is *positive* if it is self-adjoint and $q(v,v) \geq 0$ for all $v \in V$. The polarization identity implies that, at least over $\mathbf{C}$, a sesquilinear form is self-adjoint if and only if $q(v,v) \in \mathbf{R}$ for all v. In particular, a positive sesquilinear form is always self-adjoint.[2]

A fundamental result is the following result called the *Cauchy-Schwarz Inequality*.

Proposition 4.6. *If q is a positive sesquilinear form on a complex vector space V, then*

$$|q(v,w)|^2 \leq q(v,v)q(w,w).$$

Proof. Suppose that $v, w \in V$. Let $\tau \in \mathbf{T}$ be such that $\tau q(v,w) = |q(v,w)|$. Then for each $t \in \mathbf{R}$, let

$$\begin{aligned}
p(t) &= q(w + t\tau v, w + t\tau v) \\
&= q(w,w) + t\bar{\tau} q(w,v) + t\tau q(v,w) + t^2 q(v,v) \\
&= q(w,w) + 2\operatorname{Re}\big(t\tau q(v,w)\big) + t^2 q(v,v) \\
&= q(w,w) + 2t|q(v,w)| + t^2 q(v,v).
\end{aligned}$$

[2]Make these assertions into exercises. Note that if q is a sesquilinear form which satisfies $q(v,v) \in \mathbf{R}$, then using the polarization identity together with the observations that $(-i)^n = i^{4-n}$, $v + i^n w = i^n(i^{4-n}v + w)$ and $q(i^n z, i^n w) = q(z,w)$, we get

$$\begin{aligned}
4\overline{q(v,w)} &= \sum_{n=0}^{3} (-i)^n q(v + i^n w, v + i^n w) \\
&= \sum_{n=0}^{3} i^{4-n} q(i^{4-n}v + w, i^{4-n}v + w) \\
&= \sum_{k=0}^{3} i^k q(w + i^k v, w + i^k v) \\
&= 4q(w,v).
\end{aligned}$$

Since p is a quadratic polynomial and since $p(t) \geq 0$, it must have negative discriminant. Thus

$$4|q(v,w)|^2 - 4q(v,v)q(w,w) \leq 0.$$

The result follows. □

Lemma 4.7 (Cauchy-Schwarz Inequality)**.** *If f is a positive linear functional on A, then*

$$f(b^*a) = \overline{f(a^*b)} \quad \textit{and} \quad |f(b^*a)|^2 \leq f(a^*a)f(b^*b)$$

for all $a, b \in A$.

Proof. The sesquilinear form $q(a,b) = f(b^*a)$ is clearly positive. Since positive sesquilinear forms are self-adjoint,[3] $f(b^*a) = \overline{f(a^*b)}$. This proves the first assertion and the second is just the usual Cauchy-Schwarz inequality (Proposition 4.6). □

Theorem 4.8. *Every positive linear functional on a C^*-algebra A is bounded. If $\{ e_\lambda \}$ is an approximate identity for A, then*

$$\|f\| = \lim_\lambda f(e_\lambda).$$

In particular, if $\mathbf{1} \in A$, then $\|f\| = f(\mathbf{1})$.

Proof. Let $\mathscr{B}_1 = \{ a \in A : \|a\| \leq 1 \}$ be the closed unit ball in A and let $\mathscr{B}_1^+ = \mathscr{B}_1 \cap A^+$. Let $m := \sup\{ f(a) : a \in \mathscr{B}_1^+ \}$.

If $m = \infty$, then we can find $a_n \in \mathscr{B}_1^+$ such that $f(a_n) \geq n$. Let $a = \sum_{n=1}^\infty a_n/n^2$. Since A^+ is closed, $a \in A^+$ and for each $N \in \mathbf{N}$,

$$a \geq b_N := \sum_{n=1}^N \frac{a_n}{n^2} \geq 0.$$

Therefore

$$f(a) \geq f(b_N) \geq \sum_{n=1}^N \frac{1}{n}.$$

Since N is arbitrary, this leads to a contradiction. Hence we must have $m < \infty$. Since each $a \in \mathscr{B}_1$ is of the form $a = b_1 - b_2 + i(b_3 - b_4)$ with each $b_j \in \mathscr{B}_1^+$, it follows that f is bounded with $\|f\| \leq 4m$.

Let $\{e_\lambda\}$ be an approximate identity for A. Since $\lambda \leq \mu$ implies $e_\lambda \leq e_\mu$ which implies $f(e_\lambda) \leq f(e_\mu)$, $\lim_\lambda f(e_\lambda)$ exists and is bounded by $\|f\|$.

[3] Cross reference needed here.

If $a \in \mathscr{B}_1$, then

$$|f(a)|^2 = \lim_\lambda |f(e_\lambda a)|^2 \leq \limsup_\lambda f(e_\lambda^2) f(a^* a). \tag{4.2}$$

But $f(a^*a) \leq \|f\|$, and since $t^2 \leq t$ on $[0,1]$, $e_\lambda^2 \leq e_\lambda$. Therefore

$$|f(a)|^2 \leq \|f\| \lim_\lambda f(e_\lambda).$$

Since $a \in \mathscr{B}_1$ is arbitrary, we have shown that $\|f\| \leq \lim_\lambda f(e_\lambda)$. □

Corollary 4.9. *If f is a postive linear functional on A and if $\{e_\lambda\}$ is an approximate identity for A, then*

$$f(a) = \overline{f(a^*)} \quad \text{and} \quad |f(a)|^2 \leq \|f\| f(a^*a) \quad \text{for all } a \in A.$$

Proof. Let $\{e_\lambda\}$ be an approximate identity in A. We apply the first assertion in Lemma 4.7 together with the fact that f is continuous to see that

$$f(a) = \lim_\lambda f(e_\lambda a) = \lim_\lambda \overline{f(a^* e_\lambda)} = \overline{f(a^*)}.$$

This proves the first assertion. For the second, we apply the Cauchy-Schwarz Inequality for positive linear functionals as in (4.2):

$$|f(a)|^2 \leq \limsup_\lambda f(e_\lambda^2) f(a^*a) \leq \lim_\lambda f(e_\lambda) f(a^*a) = \|f\| f(a^*a).$$ □

Definition 4.10. A positive linear functional of norm one on A is called a *state* on A. The collection of states is denoted by $\mathcal{S}(A)$.

Remark 4.11. Notice that a positive linear functional on a *unital* C^*-algebra A, is a state if and only if $f(\mathbf{1}) = 1$. As the next result will show, a linear functional of norm one is a state if and only if $f(\mathbf{1}) = 1$.

Proposition 4.12. *Suppose that f is a bounded linear functional on A such that*

$$\|f\| = \lim_\lambda f(e_\lambda)$$

for some approximate identity $\{e_\lambda\}$. Then f is positive.

Proof. The first step is to see that such a f must be "real" in the sense that $f(A_{s.a.}) \subset \mathbf{R}$. Let $a \in A_{s.a.}$ with $\|a\| \leq 1$. Let $f(a) = \alpha + i\beta$. By replacing a by $-a$ if necessary, we can assume that $\beta \geq 0$.

For each $n \in \mathbf{N}$ we can choose λ_n such that $\lambda \geq \lambda_n$ implies that $\|e_\lambda a - a e_\lambda\| < \frac{1}{n}$. Then if $\lambda \geq \lambda_n$ we have

$$\|ne_\lambda - ia\|^2 = \|n^2 e_\lambda^2 + a^2 - in(e_\lambda a - a e_\lambda\| \leq n^2 + 2.$$

On the other hand, for any $n \in \mathbf{N}$,

$$\begin{aligned}\lim_\lambda |f(ne_\lambda - ia)|^2 &= \lim_\lambda \bigl|nf(e_\lambda) - i\alpha + \beta\bigr|^2 = \bigl|n\|f\| - i\alpha + \beta\bigr|^2 \\ &= \bigl(n\|f\| + \beta\bigr)^2 + \alpha^2.\end{aligned}$$

Since f is bounded, $\bigl|f(ne_\lambda - ia)\bigr|^2 \leq \|f\|^2 \|ne_\lambda - ia\|^2 \leq \|f\|^2(n^2 + 2)$ for all $\lambda \geq \lambda_n$. Thus, for each $n \in \mathbf{N}$, we must have

$$\begin{aligned}\|f\|^2(n^2+2) &\geq \lim_\lambda |f(ne_\lambda - ia)|^2 \\ &= (n\|f\| + \beta)^2 + \alpha^2 \\ &= n^2\|f\|^2 + 2n\beta\|f\| + \beta^2 + \alpha^2\end{aligned}$$

This can hold for all n only if $\beta = 0$. That is, f is "real".

Now suppose that $a \geq 0$ and $\|a\| \leq 1$. Then $e_\lambda - a \in A_{s.a.}$ and I claim that $\|e_\lambda - a\| \leq 1$. To see this, note that in A^1, $\mathbf{1}_A - (e_\lambda - a) \geq 0$ and $\mathbf{1}_A + (e_\lambda - a) \geq 0$. Now the claim follows from the functional calculus (applied to $(e_\lambda - a)$ — see E 2.3.3).

Then, since $f(A_{s.a.}) \subset \mathbf{R}$, we must have

$$f(e_\lambda - a) \leq \|f\| \quad \text{for all } \lambda.$$

Taking limits shows that $f(a) \geq 0$. This suffices. □

The next theorem is fundamental. It says that all postive linear functionals arise as in Example 4.5. As we will see, it has many uses.

Theorem 4.13 (The Gelfand-Naimark-Segal Construction)**.** *Suppose that f is a postive linear functional on a C^*-algebra A. Then there is a representation π of A with cyclic vector $h \in \mathcal{H}_\pi$ such that $\|h\|^2 = \|f\|$ and*

$$f(a) = \bigl(\pi(a)h \mid h\bigr) \quad \textit{for all } a \in A.$$

Remark 4.14. We call π the GNS-representation associated to f.

Proof. Let
$$N = \{\, a \in A : f(a^*a) = 0 \,\}.$$
The Cauchy-Schwarz Inequality implies that
$$N = \{\, a \in A : f(b^*a) = 0 \text{ for all } b \in A \,\}.$$
Therefore N is not only a vector subspace of A, but a left ideal as well. In particular, A/N is a vector space. We let $\xi : A \to A/N$ be the quotient map. We have chosen N so that
$$\big(\xi(a) \mid \xi(b)\big) := f(b^*a)$$
is a well-defined inner product on A/N. Therefore we can complete A/N to a Hilbert space $\mathcal{H}$[4].

Since N is a left-ideal in A, for each $a \in A$ we can define an operator $\pi_0(a)$ on A/N by
$$\pi_0(a)\xi(b) := \xi(ab).$$
Furthermore, π_0 is a homomorphism of A into the linear operators on A/N.

Since $a^*a \le \|a\|^2\mathbf{1}_{\widetilde{A}}$, we have $b^*a^*ab \le \|a\|^2 b^*b$ for any $a, b \in A$ (see E 2.3.2). Therefore
$$\begin{aligned}\|\pi_0(a)\xi(b)\|^2 &= \|\xi(ab)\|^2\\ &= f(b^*a^*ab)\\ &\le \|a\|^2 f(b^*b)\\ &= \|a\|^2\|\xi(b)\|^2.\end{aligned}$$
It follows that $\pi_0(a)$ is bounded and extends to an operator $\pi(a)$ on $\mathcal{H}$. Clearly, $\pi : A \to B(\mathcal{H})$ is an algebra homomorphism. Since
$$\big(\pi_0(a)\xi(b) \mid \xi(c)\big) = f(c^*ab) = \big(\xi(b) \mid \pi_0(a^*)\xi(c)\big),$$
we must have $\pi(a)^* = \pi(a^*)$. Thus π is a representation.

Now let $\{e_\lambda\}$ be an approximate identity in A. If $\mu \le \lambda$, then $e_\lambda - e_\mu \ge 0$ and $(\mathbf{1}_{\widetilde{A}} - (e_\lambda - e_\mu) \ge 0$. Thus Lemma 2.34 on page 40 implies that $\|e_\lambda - e_\mu\| \le 1$. Thus, $(e_\lambda - e_\mu)^2 \le e_\lambda - e_\mu$ and
$$\|\xi(e_\lambda) - \xi(e_\mu)\|^2 = f\big((e_\lambda - e_\mu)^2\big) \le f(e_\lambda - e_\mu).$$

[4] Cross reference needed here.

Since $\lim_\lambda f(e_\lambda) = \|f\| = \lim_\mu f(e_\mu)$, $\{\xi(e_\lambda)\}$ is a Cauchy net in $\mathcal{H}$[5]. Hence, there is a $h \in \mathcal{H}$ such that $\xi(e_\lambda) \to h$.

This means that

$$\pi(a)h = \lim_\lambda \pi_0(a)\xi(e_\lambda) = \lim_\lambda \xi(ae_\lambda).$$

But $a \mapsto \xi(a)$ is continuous because f is: $\|\xi(a) - \xi(b)\|^2 = f\big((b-a)^*(b-a)\big)$. So we can conclude that $\pi(a)h = \xi(a)$. Therefore h is a cyclic vector for π and

$$\big(\pi(a^*a)h \mid h\big) = \big(\pi(a)h \mid \pi(a)h\big) = \big(\xi(a) \mid \xi(a)\big) = f(a^*a).$$

By linearity, $f(a) = \big(\pi(a)h \mid h\big)$ for all $a \in A$.

Since π has a cyclic vector, it is definitely nondegenerate, so $\pi(e_\lambda)h \to h$ (see E 3.1.3). Thus

$$\|f\| = \lim f(e_\lambda) = \lim_\lambda \big(\pi(e_\lambda)h \mid h\big) = \|h\|^2. \qquad \square$$

Remark 4.15. It follows from the last paragraph of the proof of Theorem 4.13 that the map of A into the Hilbert space $\mathcal{H}$ of the GNS-representation associated to f is continuous with dense range.

Lemma 4.16. *Suppose that A is a C^*-algebra and that a is a nonzero normal element of A. Then there is a state $f \in \mathcal{S}(A)$ such that $|f(a)| = \|a\|$. In particular, for any $a \in A$, there is a $f \in \mathcal{S}(A)$ such that $f(a^*a) = \|a\|^2$.*

Proof. Suppose that a is a nonzero normal element. Let $B = C^*\big(\{\mathbf{1}, a\}\big) \subset \widetilde{A}$. Since $B \cong C\big(\sigma(a)\big)$ and since $\sigma(a)$ is compact, there is a complex homomorphism $h \in \Delta(B)$ such that $|h(a)| = \|\hat{a}\|_\infty = \|a\|$. (Let h correspond to evaluation at $\lambda \in \sigma(a)$ with $|\lambda| = \rho(a)$.) By the Hahn Banach Theorem, there is a norm one extension, τ, of h to $\widetilde{A}$. Since $\|\tau\| = 1 = \tau(\mathbf{1})$, τ is a state by Proposition 4.12 on page 76. Let f be the restriction of τ to A. Then f is still positive and $\|f\| \leq 1$. Since $|f(a)| = \|a\|$, $\|f\| = 1$, and f is a state. $\square$

Definition 4.17. A linear map $P : A \to B$ is called positive if $P(A^+) \subset B^+$. A positive map is called *faithful* if $a \geq 0$ and $P(a) = 0$ implies that $a = 0$.

Remark 4.18. At this point, we don't have much use for positive linear maps. But it should be noted that any $*$-homomorphism is positive. Thus a $*$-homomorphism φ is faithful exactly when it is injective: $\varphi(a) = 0$ if and only if $\varphi(a^*a) = 0$.

[5]Cross reference needed here.

Theorem 4.19 (The Gelfand-Naimark Theorem)**.** *Every C^*-algebra has a faithful nondegenerate representation.*

Proof. By Lemma 4.16, for each $a \in A \setminus \{0\}$, there is a state f_a such that $f_a(a^*a) = \|a\|^2$. Let π_a be the corresponding GNS-representation (Theorem 4.13 on page 77) with cyclic vector h_a. Then $\pi_a(a) \neq 0$ and

$$\pi := \bigoplus_{\substack{a\in A \\ a\neq 0}} \pi_a$$

is a faithful representation of A. Furthermore, π is nondegenerate since each π_a is. $\square$

Although the above Gelfand-Naimark Theorem is very satisfying, it has its drawbacks in its present form. The first is that the representation presented in the proof is hopelessly large. Another defect is that is doesn't tell us much about the existence of irreducible representations. We will take care of these complaints in the next section.

Exercises

E 4.1.1. Show that the completion of an inner product space is a Hilbert space. That is, show that if $\bigl(X, \langle \cdot\,,\cdot \rangle\bigr)$ is an inner product space, then there is a Hilbert space $\mathcal{H}$ and an isometric map ι of X onto a dense subspace of $\mathcal{H}$ such that $\bigl(\iota(x) \mid \iota(y)\bigr) = \langle x\,,y\rangle$. (Hint: start by letting $\mathcal{H} = \overline{X}$ as in Remark 1.4 on page 2.)

E 4.1.2. Suppose that A is a C^*-algebra and that f is a state on A. Show that we get a state g on A^1 extending f via $g((a,\lambda)) = f(a) + \lambda$. (Hint: compute $g\bigl((a,\lambda)^*(a,\lambda)\bigr)$ and use Corollary 4.9.)

—Answer on page 140

4.2 Pure States

A subset S of a vector space V is called *convex* if $v, w \in S$ imply that $tv+(1-t)w \in S$ for all $t \in [0,1]$. A point $z \in S$ is called an *extreme point* if whenever $z = tv+(1-t)w$ for $v, w \in S$ and $t \in (0,1)$ implies $z = v = w$.

Example 4.20. Ignoring just for the moment our fixation on complex vector spaces, let V be the real vector space $\mathbf{R}^2$. The convex sets $S_1 = \{ (x,y) \in \mathbf{R}^2 : |x|+|y| \leq 1 \}$ and $S_2 = \{ (x,y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1 \}$ are drawn in Figure 4.1. Note that S_1 has just four extreme points: $(-1,0)$, $(0,1)$, $(1,0)$ and $(0,-1)$. On the other hand, the extreme points of S_2 consist of the entire unit circle.

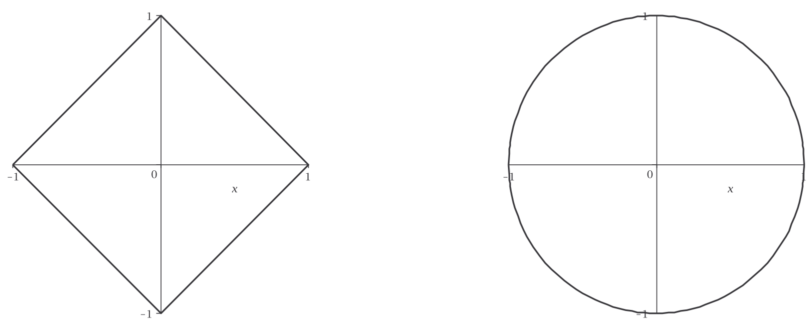


Figure 4.1: The Sets S_1 and S_2

Since the intersection of convex sets is convex, any subset S in a vector space is contained in a smallest convex set containing S called the *convex hull* of S: $\operatorname{conv}(S)$. If V is a normed vector space (or more generally a topological vector space), then the closure of a convex set is easily seen to be convex. The *closed convex hull* of S is just the closure of $\operatorname{conv}(S)$.

Our interest in convex sets starts with the following observation.

Lemma 4.21. *If A is a C^*-algebra, then the set $\mathcal{S}(A)$ of states of A is a convex subset of the unit ball of A^*.*

Proof. Suppose f_1 and f_2 are states and that $g = tf_1 + (1-t)f_2$ with $t \in [0,1]$. Then g is clearly a positive linear functional of norm at most 1. On the other hand, if $\{e_\lambda\}$ is an approximate identity for A, then by Theorem 4.8 on page 75 we have $\lim_\lambda f_i(e_\lambda) = 1$. Hence

$$\lim_\lambda g(e_\lambda) = t \lim_\lambda f_1(e_\lambda) + (1-t) \lim_\lambda f_2(e_\lambda) = 1.$$

Therefore $\|g\| = 1$ and g is state. It follows that $\mathcal{S}(A)$ is convex as claimed. $\square$

Since we especially want to study the extreme points of $\mathcal{S}(A)$, it is important to know that there are extreme points. For this, we will appeal to the Krein-Milman Theorem (see [Ped89, Theorem 2.5.4] or [Con85, Theorem V.7.4]).

Theorem 4.22. *If X is a Banach space, then every nonempty weak-$*$ compact convex subset C of X^* has an extreme point. In fact, C is the closed convex hull of its extreme points.*

Definition 4.23. If A is a C^*-algebra, then an extreme point of $\mathcal{S}(A)$ is called a *pure state*. The set of pure states is denoted by $\mathcal{P}(A)$.

Theorem 4.24. *Suppose that $\pi \in \operatorname{Rep}(A)$ and that h is a unit vector in $\mathcal{H}_\pi$ which is cyclic for π. Then*

$$f(a) := \big(\pi(a)h \mid h\big) \quad \text{for all } a \in A$$

defines a state on A which is a pure state if and only if π is irreducible.

Proof. Since h is a unit vector, it is clear that f is a state. Assume that $f \in \mathcal{P}(A)$. Suppose that $\pi \notin \operatorname{Irr}(A)$. Then there is a nontrivial projection $P \in \pi(A)'$ (Proposition 3.6 on page 50). If $Ph = 0$, then $P\pi(a)h = \pi(a)Ph = 0$ for all $a \in A$. Since $[\pi(A)h] = \mathcal{H}_\pi$, this would imply that $P = 0$. Similarly, $P^\perp h := (I - P)h \neq 0$.

Since $1 = (h \mid h) = (Ph \mid h) + (P^\perp h \mid h) = \|Ph\|^2 + \|P^\perp h\|^2$, we must have $0 < \|Ph\|^2 < 1$. Let $t = \|Ph\|^2$. Then we can define

$$g_1(a) := t^{-1}\big(\pi(a)h \mid Ph\big) \quad \text{and} \quad g_2(a) = (1-t)^{-1}\big(\pi(a)h \mid P^\perp h\big).$$

Since $g_1(a) = t^{-1}\big(\pi(a)Ph \mid Ph\big)$, g_1 is a positive linear functional and

$$\lim_\lambda g_1(e_\lambda) = \lim_\lambda t^{-1}\big(\pi(e_\lambda)Ph \mid Ph\big) = t^{-1}\|Ph\|^2 = 1.$$

It follows that g_1 is a state, and a similar argument shows that g_2 is as well.

But,

$$f = tg_1 + (1-t)g_2.$$

Since f is pure, we must have $g_1 = f$. But then

$$t^{-1}\big(\pi(a)h \mid Ph\big) = \big(\pi(a)h \mid h\big) \quad \text{for all } a \in A.$$

Therefore

$$\big(\pi(a)h \mid (t^{-1}P - I\big)h) = 0 \quad \text{for all } a \in A.$$

But then $(t^{-1}P - I)h = 0$ and $(t^{-1}P - I) \in \pi(A)'$. Just as above, this forces $t^{-1}P - I = 0$ which contradicts our choice of P. Hence π must be irreducible.

Now suppose that $\pi \in \mathrm{Irr}(A)$, and that $f = tg_1 + (1-t)g_2$ with $t \in (0,1)$. It is an exercise (E 4.2.1) to see that we get a well-defined positive sesquilinear form on $\mathcal{H}_0 = \operatorname{span}\{\, \pi(a)h : a \in A \,\}$ given by

$$[\pi(a)h, \pi(b)h] := tg_1(b^*a). \tag{4.3}$$

Moreover,

$$[\pi(a)h, \pi(a)h] = tg_1(a^*a) \le f(a^*a) = \|\pi(a)h\|^2.$$

Thus, by Cauchy-Schwarz,

$$[\pi(a)h, \pi(b)h] \le [\pi(a)h, \pi(a)h]^{\frac{1}{2}} [\pi(b)h, \pi(b)]^{\frac{1}{2}} \le \|\pi(a)h\| \|\pi(b)h\|.$$

Since $[\pi(A)h] = \mathcal{H}$, $[\cdot,\cdot]$ extends to a bounded sequilinear form on $\mathcal{H}$. Thus there is a $R \in B(\mathcal{H})$ such that

$$[\pi(a)h, \pi(b)h] = \big(\pi(a)h \mid R\pi(b)h\big) \quad \text{for all } a, b \in A$$

(see E 4.2.4). But

$$\begin{aligned} \big(\pi(b)h \mid R\pi(a)\pi(c)h\big) &= [\pi(b)h, \pi(ac)h] \\ &= tg_1(c^*a^*b) \\ &= [\pi(a^*b)h, \pi(c)h] \\ &= \big(\pi(a^*b)h \mid R\pi(c)h\big) \\ &= \big(\pi(b)h \mid \pi(a)R\pi(c)h\big). \end{aligned}$$

It follows that $R \in \pi(A)'$. Therefore $R = \alpha I$ for some $\alpha \in \mathbf{C}$. Therefore,

$$\begin{aligned} tg_1(a) &= \lim_\lambda tg_1(e_\lambda a) \\ &= \lim_\lambda [\pi(a)h, \pi(e_\lambda)h] \\ &= \lim_\lambda \big(\pi(a)h \mid R\pi(e_\lambda)h\big) \end{aligned}$$

which, since $\pi(e_\lambda)h \to h$ by E 3.1.3, is

$$\begin{aligned} &= \big(\pi(a)h \mid Rh\big) \\ &= \bar{\alpha}\big(\pi(a)h \mid h\big) \\ &= \bar{\alpha} f(a). \end{aligned}$$

Since $t = \lim_\lambda tg_1(e_\lambda) = \lim_\lambda \bar{\alpha} f(e_\lambda)$, we have $t = \bar{\alpha}$. Thus $g_1 = f$, and therefore $g_2 = f$ as well. That is, f is a pure state. This completes the proof. $\square$

Theorem 4.25. *If A is a C^*-algebra and if a is a nonzero normal element of A, then there is a pure state $f \in \mathcal{P}(A)$ such that $|f(a)| = \|a\|$.*

Proof. Let $M = \{ f \in \mathcal{S}(A) : |f(a)| = \|a\| \}$. It follows from Lemma 4.16 that M is nonempty. Furthermore, M is a weak-$*$ closed convex subset of the unit ball of A^*. Hence M is compact and it is a consequence of the Krein-Milman Theorem (Theorem 4.22 on page 82) that M has an extreme point g.

To see that g is pure, we need to show that it is also an extreme point in $\mathcal{S}(A)$. To this end, suppose that $g = tg_1 + (1-t)g_2$ with each $g_i \in \mathcal{S}(A)$ and $t \in (0,1)$. Then on the one hand, each

$$|g_i(a)| \leq \|a\|,$$

while on the other,

$$\|a\| = |g(a)| = |tg_1(a) + (1-t)g_2(a)| \leq t|g_1(a)| + (1-t)|g_2(a)| \leq \|a\|.$$

Since $t \in (0,1)$, we must have $|g_i(a)| = \|a\|$ for $i = 1, 2$. Therefore $g_i \in M$, and we must have $g = g_1 = g_2$. Therefore g is a pure state as required. □

We can now show that a C^*-algebra has lots of irreducible representations — enough to separate points and even determine the norm.

Corollary 4.26. *If A is a C^*-algebra and $a \in A$, then there is an irreducible representation $\pi \in \operatorname{Irr}(A)$ such that $\|\pi(a)\| = \|a\|$.*

Proof. If $a \neq 0$, use Theorem 4.25 to find a $f \in \mathcal{P}(A)$ such that $f(a^*a) = \|a\|^2$. Let π be the associated GNS-representation with unit cyclic vector h. Then π is irreducible by Theorem 4.24 on page 82 and

$$\|\pi(a)h\|^2 = \big(\pi(a^*a)h \mid h\big) = f(a^*a) = \|a\|^2$$

implies that $\|\pi(a)\| \geq \|a\|$. This suffices. □

Now we can set about resolving our complaint about the size the representation presented in the proof of Theorem 4.19 on page 80.

Definition 4.27. A subset $F \subset \mathcal{S}(A)$ is called *faithful* on A if whenever we are given $a \geq 0$ such that $f(a) = 0$ for all $f \in F$, then it follows that $a = 0$.

Notice that $\mathcal{S}(A)$ itself is faithful by Lemma 4.16 on page 79 and $\mathcal{P}(A)$ is faithful by Theorem 4.25.

We introduce the following notation. For $f \in \mathcal{S}(A)$, let π_f the associated GNS-representation with (unit) cyclic vector h_f, If $F \subset \mathcal{S}(A)$, then let

$$\pi_F := \bigoplus_{f \in F} \pi_f.$$

Proposition 4.28. *Let $F \subset \mathcal{S}(A)$ be a faithful family of states on A. Then π_F is a faithful representation of A on $B(\mathcal{H}_F)$.*

Proof. Let $a \in \ker \pi_F$. Then $\pi_f(a^*a) = 0$ for all $f \in F$. But then $f(a^*a) = \big(\pi(a^*a)h_f \mid h_f\big) = 0$ for all $f \in F$ and $a^*a = 0$. But then $a = 0$ and π_F is faithful. □

Corollary 4.29. *Suppose that A is a separable C^*-algebra. Then A has a faithful cyclic representation on a separable Hilbert space.*

Proof. If A is separable, then so is $\widetilde{A}$. Hence we may assume that A is unital. Since A is separable, $A_1^* := \{ f \in A^* : \|f\| \le 1 \}$ is compact and second countable in the weak-$*$ topology (see E 1.3.1.3.2). Therefore $\mathcal{S}(A)$ is separable. Let $\{f_i\}_{i=1}^{\infty}$ be a countable weak-$*$ dense subset of $\mathcal{S}(A)$. Then it is immediate that $\{f_i\}$ is faithful on A. Then if

$$f(a) := \sum_{n=1}^{\infty} 2^{-n} f_n(a), \tag{4.4}$$

it is an exercise (E 4.2.5) to see that (4.4) defines a faithful state on A. It follows that π_f is a faithful (cyclic) representation. As noted in Remark 4.15 on page 79, $a \mapsto \xi(a)$ is a continuous map of A onto a dense subspace of $\mathcal{H}_{\pi_f}$. Hence, $\mathcal{H}_{\pi_f}$ is separable. □

Corollary 4.30. *If A is a finite-dimensional C^*-algebra, then there are positive integers $n_1, \dots, n_m$ such that*

$$A \cong \bigoplus_{k=1}^{m} M_{n_k}.$$

Sketch of the Proof. As in the proof of the Corollary 4.29, we construct a faithful cyclic representation π_f. Since $a \mapsto \xi(a)$ is a continuous *linear* map of A onto a dense subspace of $\mathcal{H}_{\pi_f}$ and since A is finite dimensional, it follows that $\mathcal{H}_{\pi_f}$ is finite dimensional. The rest follows easily from Theorem 3.33 on page 64 applied to the identity representation of $\pi_f(A)$ in $\mathcal{K}(\mathcal{H}_{\pi_f}) = B(\mathcal{H}_{\pi_f}) \cong M_{\dim(\mathcal{H}_{\pi_f})}$. □

Exercises

E 4.2.1. Show that there is a well-defined sesquilinear form satisfying (4.3). (Hint: notice that if $\pi(a)h = 0$, then $g_1(a^*a) = 0$ and use Cauchy-Schwarz in the form of Lemma 4.7 on page 75.)

E 4.2.2. Recall that a set C in a vector space X is called convex if $x, y \in C$ and $\lambda \in [0,1]$ implies that $\lambda x + (1-\lambda)y \in C$.

(a) Suppose that $x_1, \dots, x_n \in X$. If $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$, then $\sum_{i=1}^n \lambda_i x_i$ is called a convex combination of the x_i. Show if C is convex, then any convex combination of elements from C belongs to C.

(b) Show that if C is a convex subset of a topological vector space X, then its closure, $\overline{C}$ is also convex.

(c) Suppose S is a subset of X. Let C be the set of convex combinations of elements of S. Show that C is the convex hull, $\operatorname{conv}(S)$, of S.

E 4.2.3. Show that in a Banach space, the closed convex hull of a compact set is compact. (Hint: show that the convex hull is totally bounded.)

—Answer on page 140

E 4.2.4. A sesquilinear form $q : \mathcal{H} \times \mathcal{H} \to \mathbf{C}$ on a Hilbert space $\mathcal{H}$ is called bounded if $|q(h,k)| \leq M\|h\|\|k\|$ for some $M \geq 0$. Show that if q is a bounded sesquilinear form as above, then there is a $R \in B(\mathcal{H})$ with $\|R\| \leq M$ such that $q(v,w) = (v \mid Rw)$. If q is positive, then show that R is a positive operator.

E 4.2.5. Show that the linear functional f defined by (4.4) is a state on A and that $\{f\}$ is separating for A. (Hint: evaluate f at $\mathbf{1}_A$.)

E 4.2.6. Give the details for the proof of Corollary 4.30 sketched above.

Chapter 5

Simple Examples

5.1 Direct Limits of C^*-Algebras

Definition 5.1. If A is a $*$-algebra, then a seminorm $\rho : \mathfrak{A} \to [0, \infty)$ is called a *C^*-seminorm* if for all $a, b \in \mathfrak{A}$, we have

(a) $\rho(ab) \leq \rho(a)\rho(b)$,

(b) $\rho(a^*) = \rho(a)$ and

(c) $\rho(a^*a) = \rho(a)^2$.

If ρ is a norm, then we call ρ a *C^*-norm* on $\mathfrak{A}$.

If ρ is a C^*-seminorm on a $*$-algebra $\mathfrak{A}$, then $N = \rho^{-1}(\{0\})$ is a self-adjoint two-sided ideal in $\mathfrak{A}$ and $\|a + N\| = \rho(a)$ is a C^*-norm on $\mathfrak{A}/N$. The completion (as in E 1.1.1 on page 8) is a C^*-algebra called the *enveloping C^*-algebra* of $(\mathfrak{A}, \rho)$.

Remark 5.2. Note that when we speak of a C^*-norm on a $*$-algebra, we are *not* asserting that $(\mathfrak{A}, \rho)$ is complete (and therefore a C^*-algebra). So while a $*$-algebra can have only one norm making it into a C^*-algebra (because the identity map has to be isometric), it is true that it can have many different C^*-norms with necessarily distinct completions.

Common examples of interesting C^*-seminorms are provided by $*$-homomorphisms of $\varphi : \mathfrak{A} \to B$ where B is a C^*-algebra. Then we set $\rho(a) := \|\varphi(a)\|$.

Example 5.3 (Reduced Group C^*-algebra). Suppose that G is a locally compact group with a Haar measure μ that is both left and right invariant. Then, using the basic properties of a Haar measure (as in [RW98, Appendix C] or [Wil07, Chap. 1]), we can make $\mathfrak{A} = C_c(G)$ into a $*$-algebra:

$$f * g(s) := \int_G f(r)g(r^{-1}s)\,d\mu(r) \quad \text{and} \quad f^*(s) = \overline{f(s^{-1})}.$$

Then, sweeping a bit of measure theory aside, we get a $*$-homomorphism $\lambda : C_c(G) \to B\big(L^2(G)\big)$ via the same formula:

$$\lambda(f)h(s) := \int_G f(r)h(r^{-1}s)\,d\mu(r).$$

(It turns out that $\|\lambda(f)h\|_2 \le \|f\|_1\|g\|_2$, so that if we wish, we can, modulo a few more measure theoretic gymnastics, replace $C_c(G)$ with $L^1(G)$.) Then $\|f\|_r := \|\lambda(f)\|$ is a C^*-norm on $C_c(G)$ called the *reduced norm.*

The enveloping C^*-algebra of $\big(C_c(G), \|\cdot\|_r\big)$ is called the *reduced group C^*-algebra* of G.

Definition 5.4. A *direct sequence* of C^*-algebras is a collection $\{(A_n, \varphi_n)\}_{n=1}^{\infty}$ where each A_n is a C^*-algebra and $\varphi_n : A_n \to A_{n+1}$ is a $*$-homomorphism. We often write

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots.$$

Example 5.5. Let $A_n = M_{2^n}$ be the $2^n \times 2^n$ complex matrices. Define $\varphi_n : M_{2^n} \to M_{2^{n+1}}$ by sending M to

$$M \oplus M := \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}.$$

Then $M_2 \xrightarrow{\varphi_1} M_4 \xrightarrow{\varphi_2} M_8 \xrightarrow{\varphi_3} \cdots$ is a direct sequence.

We want to build a "direct limit" C^*-algebra out of a direct sequence. Although our construction will be an appropriate enveloping algebra, it is best to think of a direct limit C^*-algebra via its universal property. Informally, a direct limit is a union of the A_i where the φ_i are meant to act like inclusions — see Example 5.8 on the next page. Formally, we proceed as follows.[1]

[1]Direct limits can be defined not just for sequences, but for any directed system in any category. (Here we are only looking at the direct sequences in the category of C^*-algebras and $*$-homomorphisms.) Although defined, a direct limit may or may not exist in a given category. The point of Theorem 5.10 on the facing page is that they do exist in the category we are interested in.

Definition 5.6. If $\{ (A_n, \varphi_n) \}$ is a direct sequence C^*-algebras, then a family $\{ \psi^n : A_n \to B \}$ of $*$-homomorphisms into a C^*-algebra B is said to be *compatible* if

$$\begin{tikzcd} A_n \arrow[rr, "\varphi_n"] \arrow[dr, "\psi^n"'] & & A_{n+1} \arrow[dl, "\psi^{n+1}"] \\ & B & \end{tikzcd} \tag{5.1}$$

commutes for all n.

Definition 5.7. A *direct limit C^*-algebra* for a direct sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

is a C^*-algebra A together with compatible $*$-homomorphisms $\varphi^n : A_n \to A$ which satisfy the following universal property: given compatible $*$-homomorphisms ψ^n into a C^*-algebra B, there is a *unique* $*$-homomorphism $\psi : A \to B$ such that

$$\begin{tikzcd} A \arrow[rr, "\psi"] & & B \\ & A_n \arrow[ul, "\varphi^n"] \arrow[ur, "\psi^n"'] & \end{tikzcd} \tag{5.2}$$

commutes for all n.

Example 5.8. Suppose that A_n is a C^*-subalgebra of a C^*-algebra A for $n \in \mathbf{N}$, that $A_n \subset A_{n+1}$, and that $\bigcup_n A_n$ is dense in A. Let φ_n be the inclusion of A_n in A_{n+1} and φ^n the inclusion of A_n in A. Then (A, φ^n) is the direct limit of (A_n, φ_n). For a very concrete example, let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space with orthonormal basis $\{e_k\}_{k=1}^{\infty}$. Let P_n be the projection onto the subspace spanned by $\{e_1, \dots, e_n\}$, and let $A_n = P_n \mathcal{K}(\mathcal{H}) P_n$ and φ_n the inclusion of A_n into A_{n+1}. Then, since $\bigcup A_n$ consists of the finite-rank operators on $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ is the direct limit of the A_n.

Remark 5.9. General nonsense implies that the direct limit, *if it exists*, is unique up to isomorphism. Therefore we will use the definite article and refer to *the* direct limit and use the notation $(A, \varphi^n) = \varinjlim (A_n, \varphi_n)$. If $n \geq m$, then we will use the notation $\varphi_{mn} : A_m \to A_n$ for the repeated composition $\varphi_{mn} := \varphi_{n-1} \circ \cdots \circ \varphi_{m+1} \circ \varphi_m$, with the understanding that $\varphi_{nn} := \mathrm{id}_{A_n}$.

Theorem 5.10. *Suppose that $\{ (A_n, \varphi_n) \}$ is a direct sequence of C^*-algebras. Then the direct limit $(A, \varphi^n) = \varinjlim (A_n, \varphi_n)$ always exists. Furthermore,*

(a) $\varphi^n(A_n) \subset \varphi^{n+1}(A_{n+1})$,
(b) $\bigcup_n \varphi^n(A_n)$ *is dense in* A,
(c) $\|\varphi^n(a)\| = \lim_k \|\varphi_{n,n+k}(a)\|$ *and*
(d) $\varphi^n(a) = \varphi^m(b)$ *if and only if for all* $\epsilon > 0$ *there is a* $k \geq \max\{\, m, n \,\}$ *such that*

$$\|\varphi_{nk}(a) - \varphi_{mk}(b)\| < \epsilon. \tag{5.3}$$

Proof. Let $B = \bigoplus_n A_n$ be the C^*-direct product (as in E 2.1.3), and let

$$\mathfrak{A} = \{\, \mathbf{a} = (a_n) \in B : a_{k+1} = \varphi_k(a_k) \text{ for all } k \text{ sufficiently large} \,\}.$$

Then $\mathfrak{A}$ is a $*$-subalgebra of B, and if $\mathbf{a} \in \mathfrak{A}$, then we eventually have $\|a_{k+1}\| \leq \|a_k\|$. Thus we can define

$$\rho(\mathbf{a}) = \lim_n \|a_n\| \quad \text{for all } \mathbf{a} \in \mathfrak{A}. \tag{5.4}$$

It is not hard to check that ρ is a C^*-seminorm on $\mathfrak{A}$ (see E 5.1.1). We let A be the enveloping C^*-algebra.

Furthermore, we can define $*$-homomorphisms $\hat{\varphi}^n : A_n \to \mathfrak{A}$ by

$$\big(\hat{\varphi}^n(a)\big)_k = \begin{cases} \varphi_{nk}(a) & \text{if } k \geq n \\ 0 & \text{otherwise.} \end{cases}$$

We then get compatible $*$-homomorphisms φ^n by composing with the natural map of $\mathfrak{A}$ into A (which need not be injective).

Since the remaining assertions are clear, it will suffice to show that (A, φ^n) has the right universal property. So, suppose that $\psi^n : A_n \to B$ are compatible $*$-homomorphisms. Since compatibility forces us to have $\psi\big(\varphi^n(a)\big) = \psi^n(a)$, the map ψ can exist only if $\varphi^n(a) = \varphi^m(b)$ implies that $\psi^n(a) = \psi^m(b)$. Let $\epsilon > 0$ and assume that $\varphi^n(a) = \varphi^m(b)$. Then there is a $k \geq \max\{n, m\}$ such that $\|\varphi_{nk}(a) - \varphi_{mk}(b)\| < \epsilon$. And then

$$\|\psi^n(a) - \psi^m(b)\| \leq \|\psi^k\big(\varphi_{nk}(a) - \varphi_{mk}(b)\big)\| \leq \|\varphi_{nk}(a) - \varphi_{mk}(b)\| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we get a well-defined map on $C := \bigcup_n \varphi^n(A_n)$, which is clearly dense in A. Since

$$\begin{aligned} \|\psi\big(\varphi^n(a)\big) &= \|\psi^n(a)\| \\ &= \|\psi^{n+k}\big(\varphi_{n,n+k}(a)\big)\| \quad \text{for any } k \geq 1, \\ &\leq \inf_k \|\varphi_{n,n+k}(a)\| = \lim_k \|\varphi_{n,n+k}(a)\| \\ &= \|\varphi^n(a)\|. \end{aligned}$$

Therefore ψ is norm decreasing and extends to all of A. We have $\psi \circ \varphi^n = \psi^n$ by construction, and ψ is uniquely determined on the dense subalgebra C, so ψ is unique. □

Remark 5.11. In most applications, the maps φ_n in a direct sequence are injective. Then it is not hard to see (see E 5.1.4) that conditions (c) and (d) in Theorem 5.10 can be replaced by

(c′) each φ^n is injective, and

(d′) $\varphi^n(a) = \varphi^m(b)$ if and only if there is a $k \geq \max\{m, n\}$ such that $\varphi_{nk}(a) = \varphi_{mk}(b)$.

Remark 5.12. We can also form the *algebraic direct limit* of $\{ (A_n, \varphi_n) \}$ (in the category of $*$-algebras and $*$-homomorphisms. It has similar properties to the C^*-direct limit, and can be exhibited as the $*$-algebra quotient $\mathfrak{A}/N$ where N is the ideal of elements seminorm 0 in $\mathfrak{A}$ in the proof of Theorem 5.10. Of course, in our fundamental Example 5.8 on page 89, the algebraic direct limit is just $\bigcup_n A_n$.

Theorem 5.13. *Suppose that $\mathscr{S}$ is a nonempty set of nonzero simple C^*-subalgebras of a A which is* directed *by inclusion.*[2] *If*

$$\bigcup\{ B : B \in \mathscr{S} \}$$

is dense in A, then A is simple.

Proof. It suffices to show that if $\pi : A \to B$ is a surjective $*$-homomorphism onto a nonzero C^*-algebra B, then π is injective. But for any $B \in \mathscr{S}$, $\pi|_B$ is either 0 or isometric. But since π is not the zero map on $\bigcup_{B\in\mathscr{S}} B$, the fact that $\mathscr{S}$ is directed forces π to be isometric on every $B \in \mathscr{S}$. Since π is then isometric on a dense subset, it is isometric (and hence injective) on all of A. □

Corollary 5.14. *If A_n is a simple for all $n \geq k$, then $\varinjlim(A_n, \varphi_n)$ is simple.*

Proof. If the direct limit is not the 0 algebra, there must be a m such that for all $n \geq m$, $\varphi^n(A_n)$ is simple and nonzero. We have $A := \varinjlim(A_n, \varphi_n) = \overline{\bigcup_n \varphi^n(A_n)} = \overline{\bigcup_{n\geq m} \varphi^n(A_n)}$, and $\mathscr{S} := \{\varphi^n(A_n)\}_{n\geq m}$ is clearly a directed family of nonzero simple C^*-algebras. Thus A is simple by the previous result. □

[2] Thus, if $B, C \in \mathscr{S}$, then there is a $D \in \mathscr{S}$ such that $B \cup C \subset D$.

Exercises

E 5.1.1. Show that the function ρ defined in (5.4) is a C^*-seminorm as claimed there.

E 5.1.2. Prove that the direct limit C^*-algebra is unique up to $*$-isomorphism as claimed in Remark 5.9 on page 89.

E 5.1.3. Let $\varphi_n : M_n \to M_{n+1}$ be the map sending a $n \times n$ complex matrix M to the $(n+1) \times (n+1)$-matrix $M \oplus 0$ (obtained by adding a row and column of zeros). Show that the direct limit $\varinjlim (M_n, \varphi_n) \cong \mathcal{K}(\ell^2)$. What is the algebraic direct limit?

E 5.1.4. Let $\{ A_n, \varphi_n \}$ be a direct sequence of C^*-algebras in which all the connecting maps φ_n are injective. Let (A, φ^n) be the direct limit.

(a) Show that $\varphi^n(a) = \varphi^m(b)$ if and only if there is a $k \geq \max\{n, m\}$ such that $\varphi_{nk}(a) = \varphi_{mk}(b)$, and

(b) show that each φ^n is injective.

E 5.1.5. Suppose that $\big(A, \{ \varphi^n \}\big)$ is the direct limit of a direct system $\{ (A_n, \varphi_n) \}$, and that $\big(B, \{ \psi^n \}\big)$ is the direct limit of another direct system $\{ (B_n, \psi_n) \}$. Suppose that there are maps $\alpha^n : A_n \to B_n$ such that the diagrams

$$\begin{tikzcd} A_n \arrow[r, "\varphi_n"] \arrow[d, "\alpha^n"'] & A_{n+1} \arrow[d, "\alpha^{n+1}"] \\ B_n \arrow[r, "\psi_n"'] & B_{n+1} \end{tikzcd}$$

commute for all n. Show that there is a unique homomorphism $\alpha : A \to B$ such that

$$\begin{tikzcd} A_n \arrow[r, "\alpha^n"] \arrow[d, "\varphi^n"'] & B_n \arrow[d, "\psi^n"] \\ A \arrow[r, "\alpha"'] & B \end{tikzcd}$$

commutes for all n.

E 5.1.6. Suppose that $(A, \varphi^n) = \varinjlim (A_n, \varphi_n)$, that each A_n is unital and that $\varphi_n(\mathbf{1}_{A_n}) = \mathbf{1}_{A_{n+1}}$. Show that A is unital. Is the result still true if we replace the assumption that $\varphi_n(\mathbf{1}_{A_n}) = \mathbf{1}_{A_{n+1}}$ with each φ_n being injective?

5.2 Projections and Traces

Definition 5.15. An element u in a unital C^*-algebra A is called a *isometry* if $u^*u = \mathbf{1}_A$. An invertible isometry is called a *unitary*.

Projections in a C^*-algebra were defined earlier (Definition 3.26 on page 61) as self-adjoint idempotents. For any $a \in A$, we let $|a|$ be the positive square root $(a^*a)^{\frac{1}{2}}$.

Lemma 5.16. *Suppose that p and q are projections in A such that $\|p-q\| < 1$. Then there is a unitary u in $\widetilde{A}$ such that $q = upu^*$ and such that $\|\mathbf{1} - u\| \leq \sqrt{2}\|p-q\|$. In fact, $u = v|v|^{-1}$ where $v = \mathbf{1} - p - q + 2qp$.*

Proof. Let $v = \mathbf{1}-p-q+2qp$. Then $v^* = \mathbf{1}-p-q+2pq$ and $v^*v = \mathbf{1}-(q-p)^2 = vv^*$. Thus v is normal.

Since $\|p-q\| < 1$, we also have $\|(p-q)^2\| < 1$. Hence $v^*v \in \operatorname{Inv}(\widetilde{A})$. Since v is normal, this means $v \in \operatorname{Inv}(\widetilde{A})$. Therefore $u := v|v|^{-1}$ is invertible and

$$u^*u = |v|^{-1}v^*v|v| = \mathbf{1}.$$

Hence u is unitary.

Now on the one hand,

$$vp = (\mathbf{1} - p - q + 2qp)p = p - p - qp + 2qp = qp$$

while on the other hand,

$$qv = q(\mathbf{1} - p - q + 2qp) = q - qp - q + 2qp = qp.$$

Thus $vp = qv$ and $pv^* = v^*q$. Thus

$$pv^*v = v^*qv = v^*vp.$$

Since p commutes with v^*v, it must also commute with $|v|$, and therefore with $|v|^{-1}$ as well. Therefore

$$up = v|v|^{-1}p = vp|v|^{-1} = qv|v|^{-1} = qu.$$

Thus $q = upu^*$ as required.

To establish the norm inequality, first note that $v^*v = \mathbf{1} - (p-q)^2$ has norm at most 1. Hence the same is true of $|v|$. Also

$$\operatorname{Re}(v) := \frac{v+v^*}{2} = \mathbf{1} - (p-q)^2 = |v|^2.$$

Since v is normal, v and v^*v commute. Thus v and $|v|^{-1}$ commute and

$$\operatorname{Re}(u) = \operatorname{Re}(v)|v|^{-1} = |v|.$$

Thus

$$\begin{aligned}\|\mathbf{1} - u\|^2 &= \|(\mathbf{1} - u^*)(\mathbf{1} - u)\| = \|2\mathbf{1} - u - u^*\| \\ &= 2\|\mathbf{1} - \operatorname{Re}(u)\| \\ &= 2\|\mathbf{1} - |v|\|\end{aligned}$$

which, since $1 - t \leq 1 - t^2$ for $t \in [0, 1]$, is

$$\begin{aligned}&\leq 2\|e - |v|^2\| \\ &= 2\|(p - q)^2\| \\ &= 2\|p - q\|^2.\end{aligned}$$

□

Lemma 5.17. *Suppose that a is a self-adjoint element in A such that $\|a^2 - a\| < \frac{1}{4}$. Then there is a projection in A such that $\|a - p\| < \frac{1}{2}$.*

Proof. We may as well work in $C^*\big(\{a\}\big) \cong C_0\big(\sigma(a) \setminus \{0\}\big)$. The norm condition on a implies that if $t \in \sigma(a) \subset \mathbf{R}$, then $|t| \neq \frac{1}{2}$ and $|t| < \frac{3}{2}$. Thus

$$S := \{\, t \in \sigma(a) : |t| > \frac{1}{2} \,\} = \{\, t \in \sigma(a) : |t| \geq \frac{1}{2} \,\}$$

is closed and open in $\sigma(a)$.[3] Therefore the indicator function $p' := \mathbb{1}_S$ is a projection in $C_0\big(\sigma(a) \setminus \{0\}\big)$. (If $S = \emptyset$, then we interpret $\mathbb{1}_S$ as the zero function.) Since $\|t - \mathbb{1}_S(t)\| < \frac{1}{2}$ for all t in the compact set $\sigma(a)$, we have $\|\operatorname{id} - p'\|_\infty < \frac{1}{2}$. Thus there is a projection p in A such that $\|p - a\| < \frac{1}{2}$. □

Definition 5.18. A positive linear functional on a C^*-algebra A is called *tracial* if $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$. A tracial linear functional is called faithful if $\tau(a^*a) = 0$ implies $a = 0$.

Lemma 5.19. *A positive linear functional τ is tracial if and only if $\tau(ab) = \tau(ba)$ for all $a, b \in A$.*

Proof. Suppose τ is tracial. If $x, y \in A_{s.a.}$, let $a = x + iy$. Then $a^*a = x^2 + y^2 + i(xy - yx)$, while $aa^* = x^2 + y^2 + i(yx - xy)$. Since $\tau(a^*a) = \tau(aa^*)$, we must have $\tau(xy - yx) = 0$. Thus $\tau(xy) = \tau(yx)$ for $x, y \in A_{s.a.}$. But if $b = z + iw$ with $z, w \in A_{s.a.}$, then $ab = xz + ixw + iyz - yw$ and $\tau(ab) = \tau(ba)$. □

[3] Sets that are both closed and open are sometimes called *clopen*.

Example 5.20. The C^*-algebra M_n of $n \times n$ complex matrices has a unique tracial state given by $(a_{ij}) \mapsto \frac{1}{n}\sum_{i=1}^n a_{ii}$.

Proof. Any two rank-one projections p and q in M_n are unitarily equivalent: $p = uqu^*$. Therefore any tracial state must send p to $\frac{1}{n}$. Since the rank-one projections span (by the Spectral Theorem), the assertion follows. □

Example 5.21. If $\dim(\mathcal{H}) = \infty$, then $\mathcal{K}(\mathcal{H})$ does not have a tracial state.

Proof. As in Example 5.20, any tracial state τ must take the same value t on any rank-one projection p. But if $\{p_1, \dots, p_n\}$ are pairwise orthogonal rank-one projections, then $q = p_1 + \cdots + q_n$ is a projection and $\tau(q) = nt \leq 1$. Thus, we must have $t = 0$. But then $\tau = 0$ which is a contradiction. □

Lemma 5.22. *Suppose that A is a unital C^*-algebra with C^*-subalgebras $\{A_n\}$ such that $\mathbf{1}_A \in A_n \subset A_{n+1}$ for all $n \geq 1$. If $A = \overline{\bigcup_n A_n}$ and each A_n has a unique tracial state, then A has a unique tracial state.*

Proof. Let τ_n be the unique tracial state on A_n. The restriction of τ_{n+1} to A_n is a tracial state on A_n so that $\tau_n = \tau_{n+1}|_{A_n}$. Thus we can define τ on $\bigcup A_n$ by $\tau(a) = \tau_n(a)$ if $a \in A_n$. Then τ has norm one and extends to a norm one linear functional on A such that $\tau(\mathbf{1}_A) = 1 = \|\tau\|$. Hence τ is a state (by Proposition 4.12 on page 76). Uniqueness is clear. □

Exercises

E 5.2.1. (See E 2.3.4.) Let A be a C^*-algebra. We call $u \in A$ a *partial isometry* if u^*u is a projection. Show that the following are equivalent.

(a) u is a partial isometry.
(b) $u = uu^*u$.
(c) $u^* = u^*uu^*$.
(d) uu^* is a projection.
(e) u^* is a partial isometry.

(Suggestion: for (b) $\implies$ (a), use the C^*-norm identity on $\|uu^*u - u\| = \|u(u^*u - 1)\|$.)

5.3 UHF Algebras

Definition 5.23. A unital C^*-algebra A is called a *UHF algebra* if A has *simple* finite-dimensional subalgebras A_n such that $\mathbf{1}_A \in A_n \subset A_{n+1}$ for all $n \geq 1$, and such that $A = \overline{\bigcup_n A_n}$.

Remark 5.24. “UHF” stands for *uniformly hyperfinite*. UHF algebras are also known as *Glimm algebras* as they played a prominent role is Glimm’s fundamental work on the classification of GCR algebras.[4]

Remark 5.25. In view of Corollary 4.30 on page 85, each A_n in Definition 5.23 must be (isomorphic to) a matrix algebra M_{n_k}. Therefore, each A_n is simple and admits a unique tracial state. Hence, every UHF algebra is simple and has a unique tracial state by Corollary 5.14 on page 91 and Lemma 5.22 on the previous page.

Lemma 5.26. *Suppose that $\varphi : M_n \to M_k$ is a unital $*$-homomorphism. Then $k = dn$ for some $d \geq 1$ and there is a unitary $U \in M_k$ such that*

$$\varphi(T) = U \begin{pmatrix} T & 0 & \cdots & 0 \\ 0 & T & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & T \end{pmatrix} U^*. \tag{5.5}$$

Proof. Consider φ as a representation of $M_n = \mathcal{K}(\mathbf{C}^n)$ into $M_k = B(\mathbf{C}^k)$. Then apply Corollary 3.34 on page 65. □

Notation such as that used in (5.5) can get fairly cumbersome and hence distracting (and not to mention hard to typeset even in LaTeX). We will adopt the following conventions to help. If $T \in M_n$ and $S \in M_m$, then $T \oplus S$ is meant to be the block diagonal matrix

$$\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}$$

in M_{n+m}. Then (5.5) becomes “$\varphi(T) = U(T \oplus \cdots \oplus T)U^*$”. We will follow Murhpy in [Mur90] and call the map $T \mapsto T \oplus \cdots \oplus T$ from M_n to M_{dn} the *canonical map*.

Let $\mathbf{Z}_+ = \{1, 2, 3, \dots\}$.[5] [6] Then, if view of Lemma 5.26, it is fairly clear that up to isomorphism we get a general UHF algebra by

[4] Cross reference needed here.

[5] Decide here and before that $\mathbf{N} = \{0, 1, 2, \dots\}$. Go back and fix.

[6] This obscure notation is to avoid any controversy over whether $\mathbf{N}$ contains 0 or not. It appears some people actually care.

(a) Picking a function $s : \mathbf{Z}_+ \to \mathbf{Z}_+$.
(b) Letting $s!(n) := s(1)s(2)\cdots s(n)$.
(c) Letting φ_n be the canonical map from $M_{s!(n)}$ to $M_{s!(n+1)}$.
(d) Letting $M_s := \varinjlim\big(M_{s!(n)}, \varphi_n\big)$.

Definition 5.27. Let $\mathbf{P} = \{2, 3, 5, \dots\}$ be the set of prime numbers. For each $s : \mathbf{Z}_+ \to \mathbf{Z}_+$, define $\epsilon_s : \mathbf{P} \to \mathbf{N} \cup \{\infty\}$ by

$$\epsilon_s(p) = \sup\{\, m \geq 0 : p^m \text{ divides some } s!(n) \,\}.$$

Example 5.28. Suppose that $s(n) = 2$ for all n. Then

$$\epsilon_s(p) = \begin{cases} \infty & \text{if } n = 2, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.29. *Suppose that s and s' are both functions from $\mathbf{Z}_+$ to itself such that $M_s \cong M_{s'}$. Then $\epsilon_s = \epsilon_{s'}$.*

Proof. Let $L : M_s \to M_{s'}$ be a $*$-isomorphism, and let τ and τ' be the tracial states on M_s and $M_{s'}$, respectively. Also let $\varphi^n : M_{s!(n)} \to M_s$ and $\theta^n : M_{s'!(n)} \to M_{s'}$ be the natural maps (as direct limits). Note that $\tau' \circ L = \tau$ by uniqueness, and that the maps φ^n and θ^m are all isometric.

By symmetry, it will suffice to show that $\epsilon_s(d) \leq \epsilon_{s'}(d)$ for all primes d.[7] For this, I claim it will suffice to see that given $n \in \mathbf{Z}_+$, there is a $m \in \mathbf{Z}_+$ such that $s!(n) \mid s'!(m)$. (To establish the claim, just note that $d^k \mid s!(n)$ then implies $d^k \mid s'!(m)$, and then $\epsilon_s(d) \leq \epsilon_{s'}(d)$.)

Let p be a rank-one projection in $M_{s!(n)}$. Since $\tau \circ \varphi^n$ is the unique tracial state on $M_{s!(n)}$, we have

$$\tau\big(\varphi^n(p)\big) = \frac{1}{s!(n)}.$$

On the other hand, $L\big(\varphi^n(p)\big)$ is a projection in $M_{s'}$. Therefore we can find self-adjoint element $a \in M_{s'!(m)}$ such that

$$\|L\big(\varphi^n(p)\big) - \theta^m(a)\| < \frac{1}{8} \quad \text{and} \quad \|L\big(\varphi^n(p)\big) - \theta^m(a^2)\| < \frac{1}{8}.$$

[7] Yes, d is an odd choice for a prime, but there are only 26 letters available and pretty soon p and q are going to be projections. So, I turned p upside down.

Then

$$\begin{aligned}\|a-a^2\| &= \|\theta^m(a)-\theta^m(a^2)\| \\ &\le \|\theta^m(a)-L\bigl(\varphi^n(p)\bigr)\| + \|L\bigl(\varphi^n(a)\bigr)-\theta^m(a^2)\| \\ &< \frac{1}{4}.\end{aligned}$$

Therefore, by Lemma 5.17, there is a projection $q \in M_{s'!(m)}$ such that $\|a-q\| < \frac{1}{2}$. Then

$$\begin{aligned}\|L\bigl(\varphi^n(p)\bigr)-\theta^m(q)\| &\le \|L\bigl(\varphi^n(p)\bigr)-\theta^m(a)\| + \|\theta^m(a)-\theta^m(q)\| \\ &< \frac{1}{8}+\frac{1}{2} < 1.\end{aligned}$$

Thus, but Lemma 5.16, $L\bigl(\varphi^n(p)\bigr)$ and $\theta^m(q)$ are unitarily equivalent. Hence

$$\tau'\bigl(\theta^m(q)\bigr) = \tau'\Bigl(L\bigl(\varphi^n(p)\bigr)\Bigr) = \tau\bigl(\varphi^n(p)\bigr) = \frac{1}{s!(n)}.$$

On the other hand, $\tau' \circ \theta^m$ is the tracial state on $M_{s'!(m)}$. This means that for some $d \ge 0$,

$$\frac{1}{s!(n)} = \tau'\bigl(\theta^m(q)\bigr) = \frac{d}{s'!(m)}.$$

In other words, $s'!(m) = ds!(n)$. This is what we wanted to show. $\square$

Corollary 5.30. *There are uncountably many nonisomorphic UHF algebras.*

Proof. Let $\mathbf{P} = \{2,3,5,\dots\}$ be the set of primes. For each $s : \mathbf{Z}_+ \to \mathbf{Z}_+$, define $\bar{s} : \mathbf{Z}_+ \to \mathbf{Z}_+$ by $\bar{s}(n) = p_n^{s(n)}$. Then $\epsilon_{\bar{s}}(p_n) = s(n)$. Therefore $\epsilon_{\bar{s}} = \epsilon_{\bar{s}'}$ if and only if $s = s'$. Since there are uncountably many $s : \mathbf{Z}_+ \to \mathbf{Z}_+$, we're done. $\square$

Remark 5.31. If there is time, we'll eventually show that $\epsilon_s = \epsilon_{s'}$ implies that $M_s \cong M_{s'}$.[8]

Exercises

E 5.3.1. Let M_s be a UHF algebra such which is not a matrix algebra. (This is automatic if $s : \mathbf{Z}^+ \to \{\,2,3,\dots\,\}$.) Show that M_s is not GCR.

[8] Cross reference needed here.

5.4 AF Algebras

Definition 5.32. A C^*-algebra is called approximately finite dimensional, or AF, if there are finite dimensional C^*-subalgebras $A_n \subset A_{n+1}$ such that $A = \overline{\bigcup_n A_n}$.

Example 5.33. We saw in Example 5.8 that $\mathcal{K}(\mathcal{H})$ is always AF if $\mathcal{H}$ is separable. More generally, any direct limit $A = \varinjlim\big(A_n, \varphi_n\big)$ with each A_n finite-dimensional is AF.

Theorem 5.34. *Suppose that A is AF. If I is an ideal in A, then both I and A/I are AF-algebras.*

Proof. Suppose each A_n is a finite-dimensional subalgebra of A with $A_n \subset A_{n+1}$ and $A = \overline{\bigcup A_n}$. Let $q : A \to A/I$ be the natural map. Then each $q(A_n)$ is a finite-dimensional subalgebra of A/I with $q(A_n) \subset q(A_{n+1})$ and $A/I = \overline{\bigcup q(A_n)}$. Thus, A/I is AF.

Let $I_n := I \cap A_n$. Let $J = \overline{\bigcup I_n}$. It suffices to see that $I = J$. Since we clearly have $J \subset I$, let $\varphi : A/J \to A/I$ be given by $\varphi(a + J) = a + I$. It will suffice to see that φ is isometric. Since $\bigcup (A_n + J)/J$ is dense in A/J, it suffices to see that φ is isometric on $(A_n + J)/J$. (Note that $(A_n + J)/J$ is just the image of A_n in A/J under the natural map.)

Consider the canonical isomorphisms

$$\psi : (A_n + J)/J \to A_n/(A_n \cap J) \quad \text{and} \quad \theta : (A_n + I)/I \to A_n/(A_n \cap I).$$

Since $A_n \cap I = A_n \cap J$, ψ and θ map into the same space. Let $\iota : (A_n + J)/I \to A/I$ be the inclusion map.

Then

$$\begin{tikzcd}
(A_n + J)/J \arrow[rr, "\varphi"] \arrow[d, "\psi"'] & & A/I \\
A_n/(A_n \cap J) = A_n/(A_n \cap I) \arrow[rr, "\theta^{-1}"'] & & (A_n + I)/I \arrow[u, "\iota"']
\end{tikzcd}$$

commutes. Therefore $\varphi|_{(A_n+J)/J} = \iota \circ \theta^{-1} \circ \psi$ is isometric. $\square$

Corollary 5.35. *A C^*-algebra A is AF if and only if $\widetilde{A}$ is AF.*

Proof. Since A is an ideal in $\widetilde{A}$, A is AF if $\widetilde{A}$ is. On the other hand, if $A = \overline{\bigcup A_n}$, then $\widetilde{A} = \overline{\bigcup C^*(A_n \cup \mathbf{1})}$, and $C^*(A_n \cup \mathbf{1})$ is finite-dimensional if A_n is (see E 5.4.1). $\square$

Remark 5.36. A C^*-algebra A is called *locally AF* if given a finite subset $a_1, \dots, a_n \in A$ and $\epsilon > 0$, then there is a finite-dimensional C^*-subalgebra B of A and $b_1, \dots, b_n \in B$ such that $\|a_i - b_i\| < \epsilon$. It is easy to see that an AF algebra is locally AF. The converse holds, but it is not so easy to show. A nice proof is given in [Dav96, Theorem III.3.4]. (The key is [Dav96, Lemma III.3.2] which is a bit too messy to include here.) This is a useful characterization. For example, it is easy to see that the inductive limit of AF algebras is locally AF, and hence AF.

Exercises

E 5.4.1. Show that if A_n is a finite-dimensional subalgebra of a unital C^*-algebra A, then $C^*(A_n \cup \mathbf{1}_A)$ is finite dimensional.

—Answer on page 140

Chapter 6

K-Theory

6.1 Matrix Algebras

Suppose that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Let P be the projection onto $\mathcal{H}_1$ and $P^\perp = I - P$ the projection onto $\mathcal{H}_2$. Then for any $h \in \mathcal{H}$, we have

$$\begin{aligned} Th &= TPh + TP^\perp h = PTPh + PTP^\perp h + P^\perp TPh + P^\perp TP^\perp h \\ &= \underbrace{T_{11}h + T_{12}h}_{\in \mathcal{H}_1} + \underbrace{T_{21}h + T_{22}h}_{\in \mathcal{H}_2}, \end{aligned}$$

where $T_{ij} : \mathcal{H}_j \to \mathcal{H}_i$ is the obvious operator. All this is a good deal more elegant in matrix notation:

$$T\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

Conversely, if $T_{ij} \in B(\mathcal{H}_j, \mathcal{H}_i)$, then

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

defines a bounded operator on $\mathcal{H}$. Note that

$$T^* = \begin{pmatrix} T_{11}^* & T_{21}^* \\ T_{12}^* & T_{22}^* \end{pmatrix}.$$

More generally, if $\mathcal{H}^{(n)} := \mathcal{H} \oplus \cdots \oplus \mathcal{H}$, then as above we obtain a $*$-isomorphism $\varphi : M_n\big(B(\mathcal{H})\big) \to B(\mathcal{H}^{(n)})$. We define the norm of an element $(T_{ij}) \in M_n\big(B(\mathcal{H})\big)$ by $\|(T_{ij})\| = \|\varphi\big((T_{ij})\big)\|$. The following lemma is an exercise.

Lemma 6.1. *Let* $(T_{ij}) \in M_n\big(B(\mathcal{H})\big)$. *Then*

$$\|T_{ij}\| \leq \|(T_{ij})\| \leq \sum_{ij} \|T_{ij}\|. \tag{6.1}$$

If A is a C^*-algebra, then $M_n(A)$ has a natural $*$-algebra structure extending that on $M_n = M_n(\mathbf{C})$. If $\varphi : A \to B$ is a $*$-homomorphism, then there is a natural $*$-homomorphism, called the inflation, $\varphi_n : M_n(A) \to M_n(B)$ which is defined by $\varphi_n\big((a_{ij})\big) := \big(\varphi(a_{ij})\big)$. (Shortly, in order to make the notation less cumbersome, we will start dropping the subscript "n" on the inflation maps and just use the same letter for the homomorphism and all its inflations. But for now, we'll be more pedantic.)

Theorem 6.2. *If A is a C^*-algebra, then $M_n(A)$ has a unique norm making it into a C^*-algebra.*

Proof. Let $\pi : A \to B(\mathcal{H})$ be a faithful nondegenerate representation. Then $\pi_n : M_n(A) \to M_n\big(B(\mathcal{H})\big) \cong B(\mathcal{H}^{(n)})$ is injective. The image is closed (using Lemma 6.1). Thus we get a C^*-norm on $M_n(A)$ by pulling back the norm on $M_n\big(B(\mathcal{H})\big)$: $\|(a_{ij})\| := \|(\pi(a_{ij})\|$.

The uniqueness assertion is C^*-nonsense: see E 2.5.1. □

Exercises

E 6.1.1. Prove Lemma 6.1.

6.2 The K_0 Group

Just as when working with ordinary matrices, if $a \in M_n(A)$ and $b \in M_m(A)$, then we write $a \oplus b$ for the obvious "block diagonal" matrix in $M_{n+m}(A)$:

$$a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Now we want to view $M_n(A)$ as sitting inside $M_{n+1}(A)$ in the obvious way — in the notation just introduced, we are identifying the $n \times n$-matrix a with the

$(n+1)\times(n+1)$-matrix $a\oplus 0_1$. (Here and in the sequel, 0_n denotes the $n\times n$ zero matrix.) Formally, we do this by forming the algebraic direct limit $M_\infty(A)$:

$$M_1(A) \xrightarrow{\cdot\oplus 0_1} M_2(A) \xrightarrow{\cdot\oplus 0_1} M_3(A) \xrightarrow{\cdot\oplus 0_1} \cdots.$$

This is just a fancy way of identifying a with $a\oplus 0_m$ for any $m\ge 1$.

Note that $M_\infty(A)$ is a $*$-algebra. We let

$$P[A] := \{\, p\in M_\infty(A) : p = p^* = p^2 \,\}.$$

We say that p and q in $P[A]$ are *equivalent*, or *Murray-von Neumann equivalent* if we want to be formal, if there is a $u\in M_\infty(A)$ such that $p = u^*u$ and $q = uu^*$. In this case, we write $p\sim q$.

Remark 6.3. If $p\sim q$ via u, then we can assume that p, q and u belong to $M_n(A)$ for some n. Then u is a partial isometry (see E 5.2.1). Thus if $p\sim q$ and $q\sim r$, then we have $p,q,r,u,v\in M_n(A)$, for some n, such that $p=u^*u$, $q=uu^*=v^*v$ and $r=vv^*$. Let $w=vu$. Then $w^*w=u^*v^*vu=u^*uu^*u=p^2=p$, and $ww^*=vuu^*v^*=vv^*vv^*=r^2=r$. Thus (Murray-von Neumann) equivalence is an equivalence relation.

Theorem 6.4. *Suppose that $p,q,r,s\in P[A]$.*

(a) *If $p\sim r$ and $q\sim s$, then $p\oplus q\sim r\oplus s$.*
(b) *$p\oplus q\sim q\oplus p$.*
(c) *If $p,q\in M_n(A)$ and if $pq=0$, so that $p+q\in P[A]\cap M_n(A)$, then $p+q\sim p\oplus q$.*

Proof. Suppose $p=u^*u$, $r=uu^*$, $q=v^*v$ and $s=vv^*$. Then $w=\begin{pmatrix} u & 0\\ 0 & v\end{pmatrix}$ seals the deal for part (a).

For part (b), let $u=\begin{pmatrix} 0 & q\\ p & 0\end{pmatrix}$. Then $u^*u=\begin{pmatrix} p & 0\\ 0 & q\end{pmatrix}=p\oplus q$, and $uu^*=\begin{pmatrix} q & 0\\ 0 & p\end{pmatrix}=q\oplus p$.

For part (c), let $u=\begin{pmatrix} p & q\\ 0_n & 0_n\end{pmatrix}\in M_{2n}(A)$. Then using $pq=0=qp$, check that $u^*u=\begin{pmatrix} p & 0\\ 0 & q\end{pmatrix}=p\oplus q$, while $uu^*=\begin{pmatrix} p+q & 0\\ 0 & 0\end{pmatrix}=(p+q)\oplus 0_n = p+q$. □

If A is *unital*, we say that $p,q\in P[A]$ are *stably equivalent* if for some $n\ge 1$ we have $p\oplus \mathbf{1}_n\sim q\oplus\mathbf{1}_n$. In this case we write $p\approx q$. It is an exercise to see that $\approx$ is an equivalence relation.

Definition 6.5. If A is a *unital* C^*-algebra, then the collection of stable equivalence classes in $P[A]$ is denoted by $K_0(A)^+$.

Theorem 6.6. *If A is a* unital *C^*-algebra, then*

$$[p] + [q] := [p \oplus q] \tag{6.2}$$

is a well-defined binary operation on $K_0(A)^+$ making the later into an cancellative abelian semigroup with identity equal to $[0]$. (Here "cancellative" means that $[p]+[r] = [q] + [r]$ implies that $[p] = [q]$.)

Proof. Theorem 6.4 implies that (6.2) is a well-defined commutative operation. It is clear that the operation is associative as is the fact that $[0]$ is an identity.

Suppose that $[p]+[r] = [q]+[r]$ for some $r \in M_m(A)$. Then for some n, $p\oplus r\oplus \mathbf{1}_n \sim q \oplus r \oplus \mathbf{1}_n$. Then

$$(\mathbf{1}_m - r) \oplus p \oplus r \oplus \mathbf{1}_n \sim (\mathbf{1}_m - r) \oplus q \oplus r \oplus \mathbf{1}_n. \tag{6.3}$$

Now Theorem 6.4 implies that $(\mathbf{1}_m - r) \oplus r \sim \mathbf{1}_m$. Again using Theorem 6.4, the left-hand side of (6.3) is equivalent to $p \oplus (\mathbf{1}_m - r) \oplus r \oplus \mathbf{1}_n \sim p \oplus \mathbf{1}_{n+m}$. Similarly, the right-hand side is equivalent to $q \oplus \mathbf{1}_{n+m}$. Hence $[p] = [q]$ as claimed. □

Lemma 6.7. *Suppose that N is a cancellative abelian semigroup. Then we can define an equivalence relation on $N \times N$ by $(m, n) \sim (k, l)$ if $m + l = k + n$. The set $\mathscr{G}(N)$ of equivalence classes is an abelian group with respect to the operation*

$$[m, n] + [m', n'] = [m + m', n + n'], \tag{6.4}$$

with identity equal to the class $[n, n]$ for any $n \in N$.

Proof. The proof is an exercise. □

Remark 6.8 (Grothendieck). The group $\mathscr{G}(N)$ is called the *Grothendieck group* of N. Note that $\varphi_N : N \to \mathscr{G}(N)$ given by $x \mapsto [x + x, x]$ is an *injective* semigroup homomorphism. Hence we can identify N with $\varphi(N)$ so that, informally at least, $\mathscr{G}(N) = \{\, m-n : m, n \in N \,\}$. Notice that if N has an identity, 0, then $\varphi_N(x) = [x, 0]$ and $[0, 0]$ is the identity of $\mathscr{G}(N)$.

Example 6.9. We have $\mathscr{G}(\mathbf{N}) \cong \mathbf{Z}$. (Serendipitously, this is valid whether or not your religion requires $0 \in \mathbf{N}$.)

Definition 6.10. If A is a unital C^*-algebra, then we define $K_0(A)$ to be the Grothendieck group of $K_0(A)^+$.

Now suppose that A and B are unital C^*-algebras and that $\varphi : A \to B$ is a $*$-homomorphism. Notice that if $p \in P[A]$, then $\varphi(p) \in P[B]$. (Really, we should write something like $\varphi_n(p)$ since we are using the inflation of φ, but it is standard to drop the subscript.) Also, $\varphi(p \oplus q) = \varphi(p) \oplus \varphi(q)$. If $[p] = [q]$ in $K_0(A)^+$, then replacing p by $p \oplus 0_m$ (or similarly for q), we can assume that $p, q \in M_n(A)$ and that $p \oplus \mathbf{1}_m \sim q \oplus \mathbf{1}_m$ in $M_{n+m}(A)$. Note that $\varphi(\mathbf{1}_m) \in P[B]$, and $\varphi(p \oplus \mathbf{1}_m) = \varphi(p) \oplus \varphi(\mathbf{1}_m) \sim \varphi(q) \oplus \varphi(\mathbf{1}_m)$. Since $K_0(B)^+$ is cancellative, $\varphi(p) \approx \varphi(q)$. It follows that φ induces a semigroup homomorphism $\varphi_* : K_0(A)^+ \to K_0(B)^+$ defined by $\varphi_*\big([p]\big) = [\varphi(p)]$. Then φ_* extends in the obvious way to a group homomorphism, also called φ_*, from $K_0(A)$ to $K_0(B)$.

Remark 6.11. To summarize in \$25 words, K_0 is a covariant functor from the category of unital C^*-algebras and $*$-homomorphisms to the category of abelian groups and group homomorphisms.

Example 6.12. Let $A = \mathbf{C}$. Then p and q in $P[\mathbf{C}]$ are equivalent if and only if they have the same rank. Since $\operatorname{rank}(p \oplus q) = \operatorname{rank}(p) + \operatorname{rank}(q)$, we have a semigroup homomorphism $r : K_0(\mathbf{C})^+ \to \{0, 1, 2, \dots\}$ which extends to a homomorphism $r : K_0(\mathbf{C}) \to \mathbf{Z}$. Since $r\big([\mathbf{1}_1]\big) = 1$, r is surjective. If $x \in \ker r$, then $x = [p] - [q]$ with p and q having the same rank. Thus $[p] = [q]$ and r is an isomorphism of $K_0(\mathbf{C})$ with $\mathbf{Z}$.

Example 6.13. Now let $A = B(\mathcal{H})$ for a separable infinite-dimensional Hilbert space $\mathcal{H}$. Note that any two infinite-rank projections in $B(\mathcal{H})$ are equivalent. Furthermore, $M_n\big(B(\mathcal{H})\big) \cong B(\mathcal{H}^{(n)}) \cong B(\mathcal{H})$. Thus if p is any projection in $P[B(\mathcal{H})]$, then $\mathbf{1}_1 \oplus p \sim \mathbf{1}_1 \approx 0$. Therefore $K_0\big(B(\mathcal{H})\big)^+ = \{0\}$. Thus $K_0\big(B(\mathcal{H})\big) = \{0\}$.

Definition 6.14. A unital C^*-algebra is called *finite* if $u \in A$ such that $u^*u = \mathbf{1}_A$ implies $uu^* = \mathbf{1}_A$. The algebra A is called *stably finite* if $M_n(A)$ is a finite C^*-algebra for all $n \geq 1$.

Remark 6.15. A finite C^*-algebra need not be stably finite[1]. It is an open question, I think, as to whether a simple finite C^*-algebra must be stably finite.

Theorem 6.16. *Every unital AF-algebra is stably finite.*

Proof. Since $M_n(A)$ is a unital AF-algebra if A is, it suffices to show that A is finite. Note that M_n is finite, so every finite-dimensional C^*-algebra is finite. In fact, every left-invertible element is invertible.

[1] Cross reference needed here.

Suppose that $u^*u = \mathbf{1}$. Using E 5.4.1, there are A_n finite dimensional C^*-subalgebras such that $\mathbf{1}_A \in A_n \subset A_{n+1}$, and such that there are $u_n \in A_n$ with $u_n \to u$. Thus we eventually have $\|\mathbf{1} - u_n^* u_n\| < 1$. So we can assume that each $u_n^* u_n$ is invertible. Since $\big((u_n^* u_n)^{-1} u_n^*\big) u_n = \mathbf{1}$, u_n is left-invertible. Hence, u_n is invertible. Let $v_n = (u_n^*)^{-1}$. Then $u = \lim_n v_n u_n^* u_n$. Since $u_n^* u_n \to \mathbf{1}$, we have $u = \lim_n v_n$. Therefore $uu^* = \lim_n v_n u_n^* = \lim_n \mathbf{1} = \mathbf{1}$. This completes the proof. $\square$

Exercises

E 6.2.1. Prove that stable equivalence is an equivalence relation on $P[A]$.

E 6.2.2. Prove Lemma 6.7.

E 6.2.3. If N is an abelian semigroup that is not cancellative, then relation specified in Lemma 6.7 may fail to be an equivalence relation on N. Instead, define $(m, n) \sim (k, l)$ if there is an $r \in N$ such that $m + l + r = k + n + r$. Show that the set of equivalence classes is a group with the same operation, (6.4), and identity. If N has cancellation, show that the two groups are isomorphic. Hence we can speak of the Grothendieck group of any abelian semigroup.

E 6.2.4. Show that passing from an abelian semigroup to its Grothendieck group (as in E 6.2.3) is a functor from abelian semigroups with semigroup homomorphisms to abelian groups with group homomorphisms.

E 6.2.5. Show that the set of Murray-von Neumann equivalence classes of projections in $P[A]$ for a unital C^*-algebra A is an abelian semigroup N and that the Grothendieck group of N is isomorphic to $K_0(A)$.

6.3 The Positive Cone

A *partial order* on a set G is a relation $\leq$ such that for all $x, y, z \in G$

(O1) $x \leq x$.

(O2) $x \leq y$ and $y \leq x$ implies $x = x$.

(O3) $x \leq y$ and $y \leq z$ implies $x \leq z$.

The pair $(G, \leq)$ is called an *partially ordered set* (or sometimes just an *ordered set*). If in addition, G is an abelian group, then $(G, \leq)$ is called an *ordered group* if we also have

(O4) $x \leq y$ implies $x + z \leq y + z$.

Definition 6.17. If G is an abelain group, then $N \subset G$ is called a *cone* in G if

(C1) $N + N \subset N$.

(C2) $G = N - N$.

(C3) $N \cap -N = \{0\}$.

Notice that cones correspond exactly to partial orders on G: $x \leq y$ if and only if $y - x \in N$. We'll write $G^+ := \{\, x \in G : x \geq 0 \,\}$.

Example 6.18. Most of the groups we work with everyday are naturally ordered groups. For example, $G = \mathbf{R}$ with the usual order. We can also order $\mathbf{Z}^k$ by letting $(\mathbf{Z}^k)^+ = \mathbf{N}^k$ where we have to commit to $\mathbf{N} = \{0, 1, 2, \dots\}$. For convenience, we will refer to this order on $\mathbf{Z}^k$ as the *usual order on* $\mathbf{Z}^k$.

Definition 6.19. If $\varphi : G_1 \to G_2$ is homomorphism and if G_1 and G_2 are ordered groups, then we say that φ is *positive* if $\varphi(G_1^+) \subset G_2^+$. If φ is a positive homomorphism, and if φ^{-1} is positive, then we say that φ is an *order isomorphism*.

In spite of our notation, which is fairly standard, $K_0(A)^+$ is not always a cone in $K_0(A)$. But we do have the following result which says it is for all AF algebras.

Theorem 6.20. *If A is a unital stably finite C^*-algebra, then $K_0(A)^+$ is a cone in $K_0(A)$ and $K_0(A)$ is a ordered group.*

Proof. Since properties (C1) and (C2) are clear, we only need to see that $K_0(A)^+ \cap -K_0(A)^+ = \{0\}$. So we suppose that $x = [p] = -[q]$ with $p, q \in P[A] \cap M_n(A)$. Then $[p \oplus q] = 0$. Let $r = p \oplus q$. Since Theorem 6.4(c) implies that $[\mathbf{1}_{2n}] = [\mathbf{1}_{2n} - r]$, we have $\mathbf{1}_{2n} \oplus \mathbf{1}_m \sim (\mathbf{1}_{2n} - r) \oplus \mathbf{1}_m$. This means that there is a $u \in M_{2n+m}(A)$ such that $u^*u = \mathbf{1}_{2n+m}$ while $uu^* = (\mathbf{1}_{2n} - r) \oplus \mathbf{1}_m$. But $M_{2n+m}(A)$ finite forces $uu^* = \mathbf{1}_{2n+m}$. Hence $r = 0$. Thus $p = q = 0$, and $x = 0$. $\square$

The ordered group $\big(K_0(A), K_0(A)^+\big)$ is almost what we need to distinguish AF algebras. To get an isomorphism invariant, we need an additional bit of information.

An ordered group (G, G^+) is called a *scaled ordered group* if possesses an *order unit* u such that $x \in G^+$ implies that there is a $k \in \mathbf{Z}_+$ such that $ku \geq x$.

In a unital stably finite C^*-algebra $u = [\mathbf{1}_A] = [\mathbf{1}_1]$ is an order unit (see E 6.3.3). Our object is to show that the scaled ordered group

$$\big(K_0(A), K_0(A)^+, [\mathbf{1}_A]\big) \tag{6.5}$$

is a complete isomorphism invariant for unital AF algebras. (The scaled group (6.5) is a special case of what is called the *Elliot invariant*.)

It is useful to pause and consider the situation for a finite-dimensional C^*-algebra. (In fact, understanding the finite-dimensional case will be critical to the proof of Elliot's Theorem.) Here and in the sequel, we will write $\{e_{ij}\}_{i,j=1}^n$ for the *standard matrix units* for M_n. Recall that

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \quad \text{and} \quad e_{ij}^* = e_{ji}.$$

Since $M_m(M_n) \cong M_m\big(B(\mathbf{C}^n)\big) \cong B\big((\mathbf{C}^n)^{(m)}\big) \cong M_{mn}$, every projection in $P[M_n]$ is unitarily equivalent to a diagonal matrix with only zeros and ones on the diagonal. Since unitary equivalence implies Murry-von Neumann equivalence which in turn implies stable equivalence, each class $[p] \in P[M_n]$ is of the form $k[e_{11}]$.

Remark 6.21. Note that if $p \approx q$ in $P[M_n]$, then $p \oplus \mathbf{1}_m \sim q \oplus \mathbf{1}_m$ and $\operatorname{rank} p = \operatorname{rank} q$. But then $p \sim q$. That is, p and q are stably equivalent in $P[M_n]$ if and only if they are Murry-von Neumann equivalent.

Suppose that $A = M_{n_1} \oplus \cdots \oplus M_{n_k}$ is a finite-dimensional C^*-algebra. Let $\{e_{ij}^l\}_{i,j=1}^{n_l}$ be the standard matrix units for M_{n_l}. Then we call the collection

$$\{\, e_{ij}^l : 1 \leq l \leq k \text{ and } 1 \leq i, j \leq n_l \,\}$$

of matrix units the *canonical basis* for A.

Let $\tau : \mathbf{Z}^k \to K_0(A)$ be the map

$$(m_1, \ldots, m_k) \mapsto m_1[e_{11}^1] + \cdots + m_k[e_{11}^k]. \tag{6.6}$$

Then τ is clearly a group homomorphism.

Theorem 6.22. *Let $A = M_{n_1} \oplus \cdots \oplus M_{n_k}$. Then the canonical map $\tau : \mathbf{Z}^k \to K_0(A)$ given in* (6.6) *is an order isomorphism when $\mathbf{Z}^k$ has the usual ordering (see Example 6.18 on the preceding page).*

Proof. Let $p \in P[A]$. Since $M_n(B \oplus C) = M_n(B) \oplus M_n(C)$, we have $p = p_1 + \cdots + p_k$ with $p_l \in P[M_{n_l}]$. But by the above remarks, each $p_l \sim k_l[e_{11}^l]$ in $P[M_{n_l}]$ and therefore in $P[A]$ as well. Therefore $\{[e_{11}^1], \dots, [e_{11}^k]\}$ generate $K_0(A)^+$ and τ is surjective.

For each l, let $\pi_l : A \to M_{n_l}$ be the quotient map. If $\tau(m_1, \dots, m_k) = m_1[e_{11}^1] + \cdots + m_k[e_{11}^k] = 0$, then $(\pi_l)_*\big(\sum m_j[e_{11}^j]\big) = m_l[e_{11}^l] = 0$. But then $e_{11}^l \oplus \cdots \oplus e_{11}^l$ is equivalent to 0. This forces $m_l = 0$. Thus τ is injective.

Since $\tau\big((\mathbf{Z}^k)^+\big) = K_0(A)^+$, both τ and τ^{-1} are positive. This completes the proof. □

Now we can restate Theorem 6.22 with some extra pizazz:

Corollary 6.23. *If A is a finite-dimensional C^*-algebra, then $K_0(A)$ is a free abelian group with a basis $\{x_1, \dots, x_k\}$ such that $K_0(A)^+ = \mathbf{N}x_1 + \cdots + \mathbf{N}x_k$.*

Since finite-dimensional C^*-algebras have a very rigid structure, $*$-homomorphisms between finite-dimensional C^*-algebras do as well. For convenience if $\mathbf{n} = (n_1, \dots, n_k)$ and $\mathbf{r} = (r_1, \dots, r_s)$ are vectors with entries in $\mathbf{Z}_+$, we let $M_{\mathbf{n}} = M_{n_1} \oplus \cdots \oplus M_{n_k}$ and $M_{\mathbf{r}} = M_{r_1} \oplus \cdots \oplus M_{r_s}$. For consistency with our notations for sums of projections, we'll view $M_{\mathbf{n}}$ as block diagonal matrices in $M_{|\mathbf{n}|}$ where naturally $|\mathbf{n}| = n_1 + \cdots + n_k$.

Given $\mathbf{n}$ and $\mathbf{r}$ as above, we'll call a matrix $M = (m_{ij}) \in M_{s\times k}(\mathbf{N})$ *admissible* if

$$\sum_{j=1}^{k} m_{ij} n_j \le r_i \quad \text{for } 1 \le i \le s.$$

Given an admissible matrix M, we can define a $*$-homomorphism $\varphi_M : M_{\mathbf{n}} \to M_{\mathbf{r}}$ as follows:

$$\varphi(T_1 \oplus \cdots \oplus T_k) = T_1' \oplus \cdots \oplus T_s',$$

where

$$T_i' = m_{i1} \cdot T_1 \oplus \cdots \oplus m_{ik} \cdot T_k \oplus 0_{d_i},$$

where

$$m_{ij} \cdot T_j = T_j \oplus \cdots \oplus T_j \quad \text{and} \quad d_i := r_i - \sum_{j=1}^{k} m_{ij} n_j.$$

The definition of φ_M does not seem very friendly until you write down some examples.

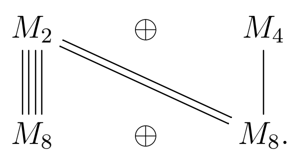
Example 6.24. Let $A = M_{(2,4)} = M_2 \oplus M_4$ and let $B = M_{(8,8)} = M_8 \oplus M_8$. Then

$$M = \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}$$

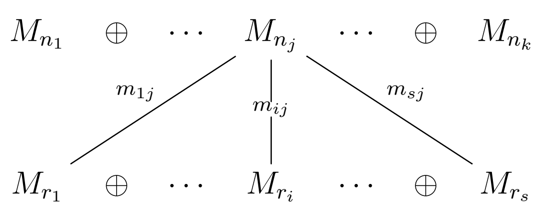
is an admissible matrix. Then φ_M is given by

$$\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \mapsto \begin{pmatrix} \begin{pmatrix} T & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & T & 0 \\ 0 & 0 & 0 & T \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & S \end{pmatrix} \end{pmatrix}$$

Even the above description is a bit overwhelming at first glance. Another way is via a sort of diagram (called a *Bratteli diagram*[2]).



More generally, for an admissible $*$-homomorphism $\varphi_M : M_{\mathbf{n}} \to M_{\mathbf{r}}$:



(6.7)

Remark 6.25. Note that an admissible $*$-homomorphism φ_M is unital if and only if $\sum_j m_{ij} n_j = r_i$ for all i.

It turns out that admissible homomorphisms are not as special as one might guess. In fact, up to a choice of orthonormal bases, every $*$-homomorphism between finite-dimensional C^*-algebras is admissible. Recall that if u is a unitary in A, then $\operatorname{Ad} u : A \to A$ is given by $(\operatorname{Ad} u)(a) = uau^*$.

[2]Cross reference needed here.

Proposition 6.26. *Let $M_{\mathbf{n}}$ and $M_{\mathbf{r}}$ be finite-dimensional C^*-algebras as above. Suppose that $\varphi : M_{\mathbf{n}} \to M_{\mathbf{r}}$ is a $*$-homomorphism. Then there is a unitary u in $M_{\mathbf{r}}$ such that $\varphi = \operatorname{Ad} u \circ \varphi_M$ for an admissible matrix $M \in M_{s\times k}(\mathbf{N})$.*

Sketch of the Proof. First, view φ as a representation of $M_{\mathbf{n}} \subset M_{|\mathbf{n}|}$ into $M_{|\mathbf{r}|} = B(\mathbf{C}^{|\mathbf{r}|})$. Then use Theorem 3.33 on page 64 to conclude that $\varphi = \bigoplus \pi^i$ with each π^i an irreducible subrepresentation equivalent to a subrepresentation of $\mathrm{id} : M_{\mathbf{n}} \to B(\mathbf{C}^{|\mathbf{n}|})$.

Secondly, conclude that $\varphi = \oplus \varphi_i$ where φ_i is a representation of $M_{\mathbf{n}}$ into $M_{r_i} = B(\mathbf{C}^{r_i})$. Use Theorem 3.33 to conclude that φ_i is equivalent to $\bigoplus_{j=1}^k m_{ij} \cdot \mathrm{id}_{M_{n_j}}$.

Conclude that there is a unitary $u_j \in M_{r_j}$ such that

$$\varphi_i(T_1 \oplus \cdots \oplus T_k) = u_j(m_{i1} \cdot T_1 \oplus \cdots \oplus m_{ik} \cdot T_k)u_j^*.$$

Then let $u = u_1 \oplus \cdots \oplus u_s$. □

Example 6.27. Suppose that $\varphi_M : M_{\mathbf{n}} \to M_{\mathbf{r}}$ is an admissible unital $*$-homomorphism as above. Then, in view of (6.7) for example,

$$(\varphi_M)_*\big([e_{11}^j]\big) = [\varphi_M(e_{11}^j)] = m_{1j}[f_{11}^1] + \cdots + m_{sj}[f_{11}^s],$$

where the $\{f_{ij}^l\}$ are the canonical basis for $M_{\mathbf{r}}$. Thus if $\tau_{\mathbf{n}} : \mathbf{Z}^k \to K_0(M_{\mathbf{n}})$ is the canonical isomorphism, then the diagram

$$\begin{tikzcd} K_0(M_{\mathbf{n}}) \arrow[r, "(\varphi_M)_*"] & K_0(M_{\mathbf{r}}) \\ \mathbf{Z}^k \arrow[u, "\cong"', "\tau_{\mathbf{n}}"] \arrow[r, "L_M"] & \mathbf{Z}^s \arrow[u, "\cong"', "\tau_{\mathbf{r}}"] \end{tikzcd} \tag{6.8}$$

commutes. (Of course, L_M denotes left-multiplication by M.)

Definition 6.28. If A and B are unital C^*-algebras, then a group homomorphism $\tau : K_0(A) \to K_0(B)$ is called *unital* if $\tau\big([\mathbf{1}_A]\big) = [\mathbf{1}_B]$.

Remark 6.29. Suppose that $\tau : K_0(M_{\mathbf{n}}) \to K_0(M_{\mathbf{r}})$ is a positive unital group homomorphism. Then, just as in diagram (6.8), we get an induced positive map $\tau' : \mathbf{Z}^k \to \mathbf{Z}^s$ which must satisfy $\tau'(n_1, \ldots, n_k) = (r_1, \ldots, r_s)$. Thus τ' is given by left-multiplication by an admissible matrix $M \in M_{s\times k}(\mathbf{Z}_+)$ satisfying $r_i = \sum_j m_{ij} n_j$ for all i.

Our next result is a version, for finite-dimensional C^*-algebras, of one of the main goals of these notes: Elliot's Theorem (Theorem 7.1 on page 117).

Theorem 6.30 (Elliot's Theorem for Finite-Dimensional Algebras)**.** *Suppose that A and B are finite-dimensional C^*-algebras.*

(a) *If $\tau : K_0(A) \to K_0(B)$ is a positive unital group homomorphism, then there is a unital $*$-homomorphism $\varphi : A \to B$ such that $\varphi_* = \tau$.*

(b) *If φ and ψ are unital $*$-homomorphisms from A to B such that $\varphi_* = \psi_*$, then there is a unitary $u \in B$ such that $\psi = \operatorname{Ad} u \circ \varphi$.*

Sketch of the Proof. By Corollary 4.30, there are $*$-isomorphisms $\sigma : A \to M_{\mathbf{n}}$ and $\sigma' : B \to M_{\mathbf{r}}$ for appropriate $\mathbf{n}$ and $\mathbf{r}$. Moreover σ_* and σ'_* are order isomorphisms. Hence we may as well assume that $A = M_{\mathbf{n}}$ and $B = M_{\mathbf{r}}$. Furthermore, if $\psi = \operatorname{Ad} u \circ \varphi$, then $\psi_* = \varphi_*$. Now the rest follows from Proposition 6.26 on the previous page, the preceding discussion and the fact that $L_M = L_{M'}$ if and only if $M = M'$. $\square$

Exercises

E 6.3.1. Provide the details for the proof of Proposition 6.26 on the preceding page.

E 6.3.2. Two projections p and q in a C^*-algebra A are called *unitarily equivalent* if there is a unitary $u \in \widetilde{A}$ such that $p = uqu^*$. Show that unitarily equivalent projections are Murry-von Neumann equivalent. (Hint: consider $w = uq$.)

E 6.3.3. Show that in a unital stably finite C^*-algebra A, $u = [\mathbf{1}_1]$ is an order unit.

6.4 Projections in Direct Limits

Lemma 6.31. *Suppose that p and q are projections in a C^*-algebra A and that there is a $u \in A$ such that $\|p - u^*u\| < 1$, $\|q - uu^*\| < 1$ and $u = qup$, then $p \sim q$.*

Proof. The first two inequalities imply that u^*u is invertible in pAp and that uu^* is invertible in qAq. Let $z = |u|^{-1}$ in pAp and set $w = uz$.

Then $w^*w = zu^*uz = z|u|^2z = pz|u|^2zp = p$.

On the other hand, $(uu^*)ww^* = uu^*uz^2u^* = u|u|^2z^2u^*$. But $u = qup$ and $p|u|^2z^2p = p$. Hence $(uu^*)ww^* = uu^*$.

But $qu = u$ and $w^*q = w^*$. Hence the invertibility of uu^* in qAq implies that $ww^* = q$. Thus $p \sim q$ as required. □

Lemma 6.32. *Suppose that $(A, \varphi^n) = \varinjlim(A_n, \varphi_n)$.*

(a) *If p is a projection in A, then there is a $n \geq 1$ and a projection $q \in A_n$ such that p is unitarily equivalent to $\varphi^n(q)$.*

(b) *If p and q are projections in A_n and if $\varphi^n(p) \sim \varphi^n(q)$, then for some $m \geq n$ we have $\varphi_{nm}(p) \sim \varphi_{nm}(q)$ in A_m.*

Proof. Let p be a projection in A. Since A is the closure of the $\varphi^n(A_n)$ we can find a sequence $\varphi^{n_k}(a_k) \to p$. Since p is self-adjoint, we can assume each a_k is as well. Since $p = p^2$, we also have $\varphi^{n_k}(a^2) \to p$. In particular, $\varphi^{n_k}(a_k - a_k^2) \to 0$. Hence there is a $m \geq 1$ and $a \in (A_m)_{s.a.}$ such that $\|\varphi^m(a) - p\| < \frac{1}{2}$ and such that $\|\varphi^m(a) - \varphi^m(a^2)\| < \frac{1}{4}$. Then for some $n \geq m$, we have $\|\varphi_{mn}(a) - \varphi_{mn}(a^2)\| < \frac{1}{4}$. In other words, $b := \varphi_{mn}(a)$ is a self-adjoint element of A_n such that $\|b - b^2\| < \frac{1}{4}$. Hence, in view of Lemma 5.17 on page 94, there is a projection $q \in A_n$ such that $\|\varphi_{mn}(a) - q\| < \frac{1}{2}$. Then

$$\begin{aligned}
\|p - \varphi^n(q)\| &\leq \|p - \varphi^m(a)\| + \|\varphi^m(a) - \varphi^n(q)\| \\
&= \|p - \varphi^m(a)\| + \|\varphi^n\big(\varphi_{mn}(a) - q\big)\| \\
&< \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}$$

Therefore Lemma 5.16 on page 93 implies that p and $\varphi^n(q)$ are unitarily equivalent. This establishes (a).

Now suppose $\varphi^n(p) \sim \varphi^n(q)$ in A. Then there is a $u \in A$ such that $\varphi^n(p) = u^*u$ and $\varphi^n(q) = uu^*$. This means that u is a partial isometry (see E 5.2.1) so that

$$u = uu^*u = u\varphi^n(p) = \varphi^n(q)u.$$

Choose $v_k \in A_{n_k}$ (with $n_k \geq n$) such that $\varphi^{n_k}(v_k) \to u$. We can replace v_k with $\varphi_{nn_k}(q)v_k\varphi_{nn_k}(p)$. Then we still have $\varphi^{n_k}(v_k) \to u$, but now $\varphi_{nn_k}(q)v_k\varphi_{nn_k}(p) = v_k$ for each k. Furthermore,

$$\varphi^n(p) = \lim_k \varphi^{n_k}(v_k^*v_k) \quad \text{and} \quad \varphi^n(q) = \lim_k \varphi^{n_k}(v_kv_k^*).$$

It follows that there is a $k \geq n$ and $v \in A_k$ such that $v = \varphi_{nk}(p)v\varphi_{nk}(q)$ and such that

$$\begin{aligned} \|\varphi^n(p) - \varphi^k(v^*v)\| &= \|\varphi^k\big(\varphi_{nk}(p) - v^*v\big)\| < 1 \quad \text{and} \\ \|\varphi^n(q) - \varphi^k(vv^*)\| &= \|\varphi^k\big(\varphi_{nk}(p) - vv^*\big)\| < 1 \end{aligned}$$

But, for example,

$$\|\varphi^k\big(\varphi_{nk}(p) - v^*v\big)\| = \lim_{r\to\infty} \|\varphi_{kr}\big(\varphi_{nk}(p) - v^*v\big)\| = \lim_{r\to\infty} \|\varphi_{nr}(p) - \varphi_{kr}(v)^*\varphi_{kr}(v)\|.$$

Hence there is a $m \geq k$ so that $w := \varphi_{km}(v)$ satisfies

$$\|\varphi_{nm}(p) - w^*w\| < 1 \quad \text{and} \quad \|\varphi_{nm}(q) - ww^*\| < 1,$$

while $w = \varphi_{km}(v) = \varphi_{km}\big(\varphi_{nk}(q)v\varphi_{nk}(p)\big) = \varphi_{nm}(p)w\varphi_{nm}(q)$. Thus we have arranged that the messy hypotheses of Lemma 6.31 on the previous page are satisfied so that we can conclude that $\varphi_{nm}(p)$ and $\varphi_{nm}(q)$ are equivalent in A_m. $\square$

Now we want to reap the consequences of Lemma 6.32 in K_0. If B is a unital subalgebra of A, and p is a projection in $P[B]$, then we'll write $[p]_B$ for the class of p in $K_0(B)$ and $[p]_A$ for the corresponding class in $K_0(A)$.

Corollary 6.33. *Suppose that A is a unital C^*-algebra with C^*-subalgebras A_n satisfying $\mathbf{1}_A \in A_n \subset A_{n+1}$ for $n \geq 1$ and such that $A = \overline{\bigcup A_n}$.*

(a) *If $p \in P[A]$, then there is a $k \geq 1$ and $q \in P[A_k]$ such that $[p]_A = [q]_A$.*

(b) *If $p, q \in P[A_k]$ and if $[p]_A = [q]_A$, then there is a $n \geq k$ such that $[p]_{A_n} = [q]_{A_n}$.*

Proof. First note that $\mathbf{1}_r \in M_r(A_n) \subset M_r(A_{n+1})$ and that $M_r(A) = \overline{\bigcup M_r(A_n)}$.

Lemma 6.32 implies that if p is a projection in $M_r(A)$, then there is a k and a projection in $M_r(A_k)$ such that p is unitarily equivalent to q in $M_r(A)$. Clearly, part (a) follows.

Lemma 6.32 also implies that if p and q are projections in $M_r(A_k)$ which are equivalent in $M_r(A)$, then they are already equivalent in some $M_r(A_n)$ with $n \geq k$. But if $[p]_A = [q]_A$, then $p \oplus \mathbf{1}_m \sim q \oplus \mathbf{1}_m$ in $M_r(A)$ for some r. Since $p \oplus \mathbf{1}_m$ and $q \oplus \mathbf{1}_m$ are projections in $M_r(A_k)$, we must have $p \oplus \mathbf{1}_m \sim q \oplus \mathbf{1}_m$ in $M_r(A_n)$ for some $n \geq k$. But then $[p]_{A_n} = [q]_{A_n}$. $\square$

The next lemma will be needed in the proof of Elliot's Theorem (Theorem 7.1). The hypotheses of stable finiteness is just used to guarantee that the K_0-group is an ordered group.

Lemma 6.34. *Suppose that A, B and C are unital stably finite C^*-algebras with A finite dimensional. Let $\tau : K_0(A) \to K_0(C)$ and $\rho : K_0(B) \to K_0(C)$ be positive homomorphisms such that $\tau\big(K_0(A)^+\big) \subset \rho\big(K_0(B)^+\big)$, Then there is a positive homomorphism $\tau' : K_0(A) \to K_0(B)$ such that*

$$\begin{tikzcd} K_0(A) \arrow[d, dotted, "\tau'"'] \arrow[dr, "\tau"] & \\ K_0(B) \arrow[r, "\rho"'] & K_0(C) \end{tikzcd}$$

commutes.

Proof. Since A is finite-dimensional, $K_0(A)$ is order isomorphic to $(Z^k, \mathbf{N}^k)$, and there is a basis for $K_0(A)$, say $\{x_1, \ldots, x_k\}$, such that $K_0(A)^+ = \mathbf{N}x_1 + \cdots + \mathbf{N}x_k$ (see Theorem 6.22 and Corollary 6.23 on page 109). Since we have assumed $\tau\big(K_0(A)^+\big) \subset \rho\big(K_0(B)^+\big)$, there are $y_1, \ldots, y_k \in K_0(B)^+$ such that $\tau(x_j) = \rho(y_j)$. But there is a unique homomorphism $\tau' : K_0(A) \to K_0(B)$ such that $\tau'(x_j) = y_j$. But then $\rho \circ \tau' = \tau$ and $\tau'\big(K_0(A)^+\big) = \mathbf{N}y_1 + \cdots + \mathbf{N}y_k \subset K_0(B)^+$, so that τ' is positive. $\square$

Chapter 7

Elliot's Theorem

The current classification program for C^*-algebras is inspired by George Elliot's notion that for many classes of C^*-algebras, their K-theory should determine the algebras up to isomorphism. This program had its genesis in Elliot's proof that unital AF-algebras were classified by their ordered K_0-groups together with their order unit. (Elliot dealt with non-unital algebras, but we have restricted to unital algebras here.) Thus the Elliot invariant of a unital AF-algebra A is the triple $\big(K_0(A), K_0(A)^+, [\mathbf{1}_A]\big)$. A unital order isomorphism $\tau : K_0(A) \to K_0(B)$ is an order isomorphism that maps $[\mathbf{1}_A]$ to $[\mathbf{1}_B]$.

7.1 Elliot's Theorem for Unital AF-Algebras

Theorem 7.1. *Suppose that A and B are unital AF-algebras and that $\tau : K_0(A) \to K_0(B)$ is a unital order isomorphism. Then there is a $*$-isomorphism $\varphi : A \to B$ such that $\varphi_* = \tau$.*

The proof will take a few pages. First we introduce some notation. We let A_n be a finite-dimensional subalgebra of A such that $\mathbf{1}_A \in A_n \subset A_{n+1}$ and such that $\bigcup A_n$ is dense in A. Similarly, we assume we have finite-dimensional subalgebras $\mathbf{1}_B \in B_n \subset B_{n+1}$ with $\bigcup B_n$ dense in B. We let $\varphi^n : A_n \to A$ and $\psi^n : B_n \to B$ be the inclusion maps. We will also write ρ for the order isomorphism τ^{-1}.

It is a consequence of Corollary 6.33 that

$$K_0(A)^+ = \bigcup \varphi^n_*\big(K_0(A_n)^+\big) \quad \text{and} \quad K_0(A) = \bigcup \varphi^n_*\big(K_0(A_n)\big). \tag{7.1}$$

Similarly,

$$K_0(B)^+ = \bigcup \psi_*^n\big(K_0(B_n)^+\big) \quad \text{and} \quad K_0(B) = \bigcup \psi_*^n\big(K_0(B_n)\big). \tag{7.2}$$

Remark 7.2. It is important to keep in mind that while the φ^n (and the ψ^m) are just inclusion maps, the induced maps φ_*^n on K-theory can have kernel: φ_*^n takes the class $[p]_{A_n}$ of a projection $p \in P[A_n]$ to the class $[p]_A$ of the same projection viewed as an element of $P[A]$.

The idea of the proof is to inductively construct subsequences

$$n_1 < m_1 < n_2 < m_2 < \cdots$$

together with $*$-homomorphisms α^k and β^k such that the diagram

$$\begin{tikzcd}
{} \arrow[r, dotted] & A_{n_k} \arrow[rr, "\varphi_{n_k n_{k+1}}"] \arrow[dr, "\alpha^k"'] & & A_{n_{k+1}} \arrow[rr, dotted] \arrow[dr, "\alpha^{k+1}"] & & {} \\
{} \arrow[ur, dotted] \arrow[rr, dotted] & & B_{m_k} \arrow[ur, "\beta^k"'] \arrow[rr, "\psi_{m_k m_{k+1}}"'] & & B_{m_{k+1}} \arrow[ur, dotted] \arrow[r, dash, dotted] & {}
\end{tikzcd} \tag{7.3}$$

commutes. We'll see that this suffices (see E 5.1.5).

The main step is the following proposition which is strictly as statement about the existence of group homomorphisms.

Proposition 7.3. *Let A, B and τ be as in the statement of Theorem 7.1. Then there are integers $1 = n_1 < m_1 < n_2 < m_2 < \cdots$ and positive unital homomorphisms $\tau^k : K_0(A_{n_k}) \to K_0(B_{m_k})$ and $\rho^k : K_0(B_{m_k}) \to K_0(A_{n_{k+1}})$ such that the diagram*

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] \\
K_0(A_{n_k}) \arrow[r, "\varphi_*^{n_k}"] \arrow[d, "\tau^k"] & K_0(A) \arrow[d, "\tau"] \\
K_0(B_{m_k}) \arrow[r, "\psi_*^{m_k}"] \arrow[d, "\rho^k"] & K_0(B) \arrow[d, "\rho"] \\
K_0(A_{n_{k+1}}) \arrow[r, "\varphi_*^{n_{k+1}}"] \arrow[d] & K_0(A) \arrow[d] \\
{} & {}
\end{tikzcd} \tag{7.4}$$

commutes, and such that $\rho^k \circ \tau^k = (\varphi_{n_k n_{k+1}})_*$ and $\tau^{k+1} \circ \rho^k = (\psi_{m_k m_{k+1}})_*$.

Proof. Let $n_1 = 1$ and choose a basis $\{x_1, \dots, x_k\}$ for $K_0(A_{n_1})$ such that $K_0(A_{n_1})^+ = \mathbf{N}x_1 + \cdots + \mathbf{N}x_k$ (Corollary 6.23). Since

$$\tau \circ \varphi_*^{n_1}\big(K_0(A_{n_1})^+\big) = \mathbf{N}\tau \circ \varphi_*^{n_1}(x_1) + \cdots + \mathbf{N}\tau \circ \varphi_*^{n_1}(x_k),$$

it follows from (7.2) that for some $m \geq n_1$ we have

$$\big(\tau \circ \varphi_*^{n_1}\big)(x_j) \in \psi_*^m\big(K_0(B_m)^+\big).$$

Hence $\tau \circ \varphi_*^{n_1}\big(K_0(A_{n_1})^+\big) \subset \psi_*^m\big(K_0(B_m)\big)$. Now Lemma 6.34 on page 115 implies that there is a positive group homomorphism $\tilde{\tau} : K_0(A_{n_1}) \to K_0(B_m)$ such that the diagram[1]

$$\begin{tikzcd}
K_0(A_{n_1}) \arrow[r, "\varphi_*^{n_1}"] \arrow[d, "\tilde{\tau}"'] \arrow[ddr, dotted, "\tau\circ\varphi_*^{n_1}"] & K_0(A) \arrow[dd, "\tau"] \\
K_0(B_m) \arrow[dr, "\psi_*^m"] & \\
& K_0(B)
\end{tikzcd}$$

commutes.

So far we have done nothing to guarantee that $\tilde{\tau}$ is unital. But if $\tilde{\tau}\big([\mathbf{1}_A]\big) = [p]_{B_m}$, then

$$\begin{aligned}
[p]_B &= \psi_*^m\big([p]_{B_m}\big) \\
&= \psi_*^m\big(\tilde{\tau}\big([\mathbf{1}_A]\big)\big) \\
&= \tau \circ \varphi_*^{n_1}\big([\mathbf{1}_A]\big) \\
&= \tau\big([\mathbf{1}_A]\big) \\
&= [\mathbf{1}_B].
\end{aligned}$$

Therefore Corollary 6.33 implies that there is a $m_1 \geq m$ such that $[p]_{B_{m_1}} = [\mathbf{1}_B]_{B_{m_1}}$.

[1] The diagrams in this proof have been "color coded": bits that are temporary constructs—and not destined to be part of (7.4)—have been drawn in blue.

Now let $\tau^1 = (\psi_{mm_1})_* \circ \tilde{\tau}$. Then τ^1 is unital and the diagram

$$\begin{tikzcd}
K_0(A_{n_1}) \arrow[r, "\varphi_*^{n_1}"] \arrow[d, "\tilde{\tau}"'] \arrow[dd, bend right=60, "\tau^1"'] & K_0(A) \arrow[dd, "\tau"] \\
K_0(B_m) \arrow[d, "(\psi_{mm_1})_*"] \arrow[dr, "\psi_*^m"] & \\
K_0(B_{m_1}) \arrow[r, "\psi_*^{m_1}"'] & K_0(B)
\end{tikzcd}$$

commutes.

Furthermore, by symmetry, there is a $n > m_1$ and a positive (unital) homomorphism $\tilde{\rho} : K_0(B_{m_1}) \to K_0(A_n)$ such that the diagram

$$\begin{tikzcd}
K_0(A_{n_1}) \arrow[r, "\varphi_*^{n_1}"] \arrow[d, "\tau^1"'] \arrow[dd, bend right=60, dotted, "\tilde{\rho}\circ\tau^1"'] & K_0(A) \arrow[d, "\tau"] \\
K_0(B_{m_1}) \arrow[r, "\psi_*^{m_1}"] \arrow[d, "\tilde{\rho}"'] & K_0(B) \arrow[dd, "\rho"] \\
K_0(A_n) \arrow[dr, "\varphi_*^n"] & \\
& K_0(A)
\end{tikzcd}$$

commutes. But we need more. We need to have $\tilde{\rho} \circ \tau^1 = (\varphi_{n_1 n})_*$.

Let $\{x_1, \ldots, x_k\}$ be our basis of $K_0(A_{n_1})$. Let $x_j = [p_j]_{A_{n_1}}$, and let $[q_j] = \tilde{\rho} \circ \tau^1(x_j)$. Now

$$\begin{aligned}
[q_j]_A &= \varphi_*^n\big([q_j]_{A_n}\big) \\
&= \varphi_*^n \circ \tilde{\rho} \circ \tau^1\big([p_j]_{A_{n_1}}\big) \\
&= \rho \circ \psi_*^{m_1} \circ \tau^1\big([p_j]_{A_{n_1}}\big) \\
&= \rho \circ \tau \circ \varphi_*^{n_1}\big([p_j]_{A_{n_1}}\big) \\
&= [p_j]_A.
\end{aligned}$$

Therefore we can apply Corollary 6.33 repeatedly to find a $n_2 \geq n$ such that $[p_j]_{A_{n_2}} =$

$[q_j]_{A_{n_2}}$ for all j. Let $\rho^1 := (\varphi_{nn_2})_* \circ \tilde{\rho}$. Thus the diagram

$$\begin{tikzcd}
K_0(A_{n_1}) \arrow[r, "\varphi_*^{n_1}"] \arrow[d, "\tau^1"'] & K_0(A) \arrow[d, "\tau"] \\
K_0(B_{m_1}) \arrow[r, "\psi_*^{m_1}"] \arrow[d, "\tilde{\rho}"'] \arrow[dd, bend right=60, "\rho^1"'] & K_0(B) \arrow[dd, "\rho"] \\
K_0(A_n) \arrow[d, "(\varphi_{nn_2})_*"'] \arrow[dr, "\varphi_*^{n}"] & \\
K_0(A_{n_2}) \arrow[r, "\varphi_*^{n_2}"'] & K_0(A)
\end{tikzcd}$$

commutes. Furthermore, by definition of ρ^1,

$$\rho^1 \circ \tau^1(x_j) = (\varphi_{nn_2})_* \circ \tilde{\rho} \circ \tau^1(x_j)$$

which, by definition of the q_j, is

$$\begin{aligned}
&= (\varphi_{nn_2})_*\big([q_j]_{A_n}\big) \\
&= [q_j]_{A_{n_2}} \\
&= [p_j]_{A_{n_2}} \\
&= (\varphi_{n_1 n_2})_*\big([p_j]_{A_{n_1}}\big) \\
&= (\varphi_{n_1 n_2})_*(x_j).
\end{aligned}$$

Therefore, $\rho^1 \circ \tau^1 = (\varphi_{n_1 n_2})_*$, and in particular, ρ^1 is unital.

Now we can proceed inductively, interchanging the roles of A and B when necessary, to complete the proof. □

Proof of Theorem 7.1. Let τ^k and ρ^k be the unital positive homomorphisms as constructed in Proposition 7.3. By Theorem 6.30 (Elliot's Theorem for finite-dimensional C^*-algebras), there are unital $*$-homomorphisms $\alpha^1 : A_{n_1} \to B_{m_1}$ and $\beta^1 : B_{m_1} \to A_{n_2}$ such that $\alpha_*^1 = \tau^1$ and $\beta_*^1 = \rho^1$.

Thus $(\beta^1 \circ \alpha^1)_* = \beta_*^1 \circ \alpha_*^1 = \rho^1 \circ \tau^1 = (\varphi_{n_1 n_2})_*$. Theorem 6.30 implies that there is a unitary $u \in A_{n_2}$ such that

$$\varphi_{n_1 n_2} = \operatorname{Ad} u \circ \beta^1 \circ \alpha^1.$$

Hence, after replacing β^1 by $\operatorname{Ad} u \circ \beta^1$, we can assume that $\beta^1 \circ \alpha^1 = \varphi_{n_1 n_2}$. Note that $(\operatorname{Ad} u)_* = \mathrm{id}_{K_0(A_{n_2})}$ so we still have $\beta^1_* = \rho^1$.

Thus we can continue inductively to get unital $*$-homomorphisms $\alpha^k : A_{n_k} \to B_{m_k}$ and $\beta^k : B_{m_k} \to A_{n_{k+1}}$ such that

$$\alpha^k_* = \tau^k, \quad \beta^k_* = \rho^k, \quad \alpha^{k+1} \circ \beta^k = \psi_{m_k m_{k+1}} \quad \text{and} \quad \beta^k \circ \alpha^k = \varphi_{n_k n_{k+1}}.$$

(In other words, we have produced the promised commutative diagram: (7.3) on page 118.)

The point is given $a \in A_{n_k}$ (and recalling that the φs and ψs are inclusion maps),

$$\begin{aligned} \alpha^{k+1}(a) &= \alpha^{k+1}\big(\varphi_{n_k n_{k+1}}(a)\big) \\ &= \alpha^{k+1}\big(\beta^k \circ \alpha^k(a)\big) \\ &= (\alpha^{k+1} \circ \beta^k)\big(\alpha^k(a)\big) \\ &= \psi_{m_k m_{k+1}}\big(\alpha^k(a)\big) \\ &= \alpha^k(a). \end{aligned}$$

Therefore we can define $\alpha : \bigcup A_{n_k} \to B$ by $\alpha(a) = \alpha^k(a)$ if $a \in A_{n_k}$. Since each α^k is norm decreasing, so is α and α extends to a $*$-homomorphism of A into B.

Similarly, we get a $*$-homomorphism $\beta : B \to A$ such that $\beta(b) = \beta^k(b)$ if $b \in B_{m_k}$.

But if $a \in A_{n_k}$, then $\beta \circ \alpha(a) = \beta^k\big(\alpha^k(a)\big) = \varphi_{n_k n_{k+1}}(a) = a$. Thus $\beta \circ \alpha = \mathrm{id}_A$. Similarly, $\alpha \circ \beta = \mathrm{id}_B$, and α is a $*$-isomorphism.

Now let $p \in P[A_{n_k}]$. We have

$$\begin{aligned} \tau\big([p]_A\big) &= \tau\big([\varphi^{n_k}(p)]_A\big) \\ &= \tau \circ \varphi^{n_k}_*\big([p]_{A_{n_k}}\big) \end{aligned}$$

which, by Proposition 7.3, is

$$\begin{aligned} &= \psi^{m_k}_* \circ \tau^k\big([p]_{A_{n_k}}\big) \\ &= \psi^{m_k}_* \circ \alpha^k_*\big([p]_{A_{n_k}}\big) \\ &= \psi^{m_k}_*\big([\alpha^k(p)]_{B_{m_k}}\big) \\ &= [\alpha^k(p)]_B \\ &= [\alpha(p)]_B \\ &= \alpha_*\big([p]_A\big). \end{aligned}$$

Therefore $\tau = \alpha_*$ on $\bigcup \varphi^{n_k}_*\big(K_0(A_{n_k})^+\big) = K_0(A)^+$. Hence $\tau = \alpha_*$. This completes the proof of Elliot's Theorem. □

Corollary 7.4. *Two unital AF-algebras are $*$-isomorphic if and only if there is a unital order isomorphism of their K_0-groups.*

Proof. If $\varphi : A \to B$ is a $*$-isomorphism, then functorality implies that φ_* is a unital order isomorphism. Elliot's Theorem provides the converse. □

7.2 Back to Glimm Algebras

In the category of sets and maps, a direct sequence is just a countable collection of sets X_i and maps φ_i:

$$X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} X_3 \xrightarrow{\varphi_3} \dots .$$

In this category, "the" direct limit is a set X together with compatible maps $\varphi^n : X_n \to X$ such that given any set Y with compatible maps $\psi^n : X_n \to Y$, there is a unique map θ such that

$$\begin{tikzcd} X \arrow[rr, "\theta"] & & Y \\ & X_n \arrow[ul, "\varphi^n"] \arrow[ur, "\psi^n"'] & \end{tikzcd}$$

commutes for all n. Of course, once we see that direct limits always exist in this category, they are appropriately unique. (This justifies the "the" above.) To see that there always is a direct limit let

$$W := \coprod_{n=1}^{\infty} X_n = \{\, (x,n) : x \in X_n \,\}$$

be the *disjoint union* of the X_n. Then there is a smallest equivalence relation $\sim$ on W such that

$$(x,n) \sim \big(\varphi_n(x), n+1\big).$$

Let $X_\infty := W/{\sim}$. It is not hard to show that X_∞, together with the maps $\varphi^n(x) := [x,n]$, is a direct limit. Furthermore, if $Y_n \subset X_n$ and $\varphi_n(Y_n) \subset Y_{n+1}$, then we can identify $\varinjlim(Y_n, \varphi_n|_{Y_n})$ with the subset $\bigcup \varphi^n(Y_n)$ of X_∞.

Example 7.5 (Direct Limits of Groups)*.* Suppose that

$$G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} G_3 \xrightarrow{\varphi_3} \dots$$

is a direct sequence of ordered (abelian) groups with each φ_i a positive homomorphism. Then I claim that $G := \varinjlim(G_n, \varphi_n)$ is an ordered group with $G^+ = \varinjlim(G_n^+, \varphi_n)$.

Sketch of the Claim. To see that G is a group, define

$$[a,n]+[b,m]=[\varphi_{nk}(a)+\varphi_{mk}(b),k]$$

for some $k\geq\max\{n,m\}$. It is not hard to see that this is well-defined and defines a group operation. To see that G^+ is a cone, then real issue is to see that

$$g\in G^+\cap -G^+\implies g=0.$$

But if g is as above, we must on the one hand have $g=[a,n]$ and $g=[-b,m]$ for some $a\in G_n^+$ and $b\in G_m^+$. Let $k=\max\{n,m\}$. Then

$$g=[\varphi_{nk}(a),k]=[-\varphi_{mk}(b),k].$$

Thus for some $p\geq k$, we must have

$$\varphi_{np}(a)=-\varphi_{mp}(b).$$

Since φ_{np} and φ_{mp} are positive maps, this entails that

$$\varphi_{np}(a)=\varphi_{mp}(b)=0.$$

Thus g is zero. □

Remark 7.6. It is not hard to see that (G,φ^n) is universal with respect to the property that given positive homomorphisms $\theta^n : G_n\to H$ which are compatible with the φ^n, then there exists a unique positive homomorphism ψ such that

$$\begin{tikzcd} G \arrow[rr, "\psi"] & & H \\ & G_n \arrow[ul, "\varphi^n"] \arrow[ur, "\theta^n"'] & \end{tikzcd}$$

commutes for all n. Of course, (G,φ^n) is unique up to order isomorphism.

Our next result is a special case of what is usually called "the continuity of K_0".[2]

Theorem 7.7. *Let (A,φ^n) be the direct limit $\varinjlim(A_n,\varphi_n)$ of unital C^*-algebras A_n and unital homomorphisms $\varphi_n : A_n\to A_{n+1}$. Then*

$$\big(K_0(A),\varphi^n_*\big)=\varinjlim\big(K_0(A_n),(\varphi_n)_*\big). \tag{7.5}$$

[2]More generally, "K_0 commutes with direct limits".

Remark 7.8 (Ordered K-Theory). If A and each of the A_n are stably finite, so that their K_0-groups are ordered groups, then since the $(\varphi_n)_*$ are positive by definition, (7.5) is an order isomorphism as discussed in Example 7.5.

For the proof of the theorem, we'll need an "upgrade" to Corollary 6.33.

Proposition 7.9. *Let (A, φ^n) be the direct limit $\varinjlim(A_n, \varphi_n)$ of unital C^*-algebras A_n and unital homomorphisms $\varphi_n : A_n \to A_{n+1}$.*

(a) *If $p \in P[A]$, then there is a $k \geq 1$ and a $q \in P[A_k]$ such that $[p]_A = \varphi^k_*\big([q]_{A_k}\big)$.*

(b) *Suppose that $p \in P[A_n]$ and $q \in P[A_m]$ are such that $\varphi^n_*\big([p]_{A_n}\big) = \varphi^m_*\big([q]_{A_m}\big)$. Then there is a $k \geq \max\{n, m\}$ such that $(\varphi_{nk})_*\big([p]_{A_n}\big) = (\varphi_{mk})_*\big([q]_{A_m}\big)$.*

Proof. Notice that for any $r \geq 1$, we have $\big(M_r(A), \varphi^n\big) = \varinjlim\big(M_r(A_n), \varphi_n\big)$, where, for example, I have also written φ_n for the inflation of φ_n to $M_r(A_n)$. Also notice that each $M_r(A_n)$ is unital with unit $\mathbf{1}^n_r$ (where $\mathbf{1}^n$ is the unit of A_n).

The point is that Lemma 6.32 on page 113 implies that any projection $p \in M_r(A)$ is unitarily equivalent to $\varphi^k(q)$ for some projection $q \in M_r(A_k)$ with $k \geq 1$. But then $[p]_A = [\varphi^k(q)]_A = \varphi^k_*\big([q]_{A_k}\big)$. This proves part (a).

For part (b), we can assume that $n \geq m$. By assumption, $\varphi^n(p) \approx \varphi^m(q)$ in $P[A]$. Thus,

$$\varphi^n(p) \oplus \mathbf{1}_s \sim \varphi^m(q) \oplus \mathbf{1}_s = \varphi^n\big(\varphi_{mn}(q)\big) \oplus \mathbf{1}_s$$

is some $M_r(A)$. Since everything in sight is unital,

$$\varphi^n\big(p \oplus \mathbf{1}^n_s\big) = \varphi^n(p) \oplus \mathbf{1}_s \sim \varphi^n\big(\varphi_{mn}(q)\big) \oplus \mathbf{1}_s = \varphi^n\big(\varphi_{mn}(q) \oplus \mathbf{1}^n_s\big).$$

Then Lemma 6.32 implies that there is a $k \geq n$ such that $\varphi_{nk}(p \oplus \mathbf{1}^n_s) \sim \varphi_{nk}\big(\varphi_{mn}(q) \oplus \mathbf{1}^n_s\big)$. But then

$$\varphi_{nk}(p) \oplus \mathbf{1}^k_s \sim \varphi_{mk}(q) \oplus \mathbf{1}^k_s.$$

But then $[\varphi_{nk}(p)]_{A_k} = [\varphi_{mk}(q)]_{A_k}$ as required. □

Proof of Theorem 7.7. First notice that since the $\{\varphi^n\}$ are compatible with the $\{\varphi_n\}$, and since K_0 is a functor, the diagrams

$$\begin{tikzcd}
K_0(A_n) \arrow[rr, "(\varphi_n)_*"] \arrow[dr, "\varphi^n_*"'] & & K_0(A_{n+1}) \arrow[dl, "\varphi^{n+1}_*"] \\
& K_0(A) &
\end{tikzcd}$$

commute for all n. This just means that maps $\{\varphi_*^n\}$ are compatible with the $\{(\varphi_n)_*\}$ and the left-hand side of (7.5) is at least an appropriate object to be a direct limit of the right-hand side.

So to complete the proof, we just need to see that $\big(K_0(A),\varphi_*^n\big)$ has the right universal property. To this end, suppose that $\theta^n : K_0(A_n) \to H$ are a family of homomorphisms compatible with the $\{(\varphi_n)_*\}$. (In the stably finite case, we would also assume the θ^n were positive.) This means that

$$\theta^k \circ (\varphi_{nk})_* = \theta^n \quad \text{for any } k \geq n.$$

But if $\varphi_*^n\big([p]_{A_n}\big) = \varphi_*^m\big([q]_{A_m}\big)$, then Proposition 7.9 implies that there is a $k \geq \max\{n, m\}$ such that

$$(\varphi_{nk})_*\big([p]_{A_n}\big) = (\varphi_{mk})_*\big([q]_{A_m}\big).$$

Then we have

$$\theta^n\big([p]_{A_n}\big) = \theta^k\big((\varphi_{nk})_*\big([p]_{A_n}\big)\big) = \theta^k\big((\varphi_{mk})_*\big([q]_{A_m}\big)\big) = \theta^m\big([q]_{A_m}\big). \tag{7.6}$$

Since Proposition 7.9 implies that

$$K_0(A)^+ = \bigcup \varphi_*^n\big(K_0(A_n)^+\big),$$

(7.6) implies that we have a well-defined semigroup homomorphism $\psi : K_0(A)^+ \to H$ determined by

$$\psi\big(\varphi_*^n\big([p]_{A_n}\big)\big) = \theta^n\big([p]_{A_n}\big). \tag{7.7}$$

Then ψ extends to a homomorphism of $K_0(A)$ into H which has the desired properties (and which is uniquely determined by (7.7)). $\square$

Let $s : \mathbf{Z}_+ \to \mathbf{Z}_+$ be any function. As in §5.3 (starting on page 96), let $s!(n) := s(1)s(2)\cdots s(n)$, and for any prime p, let

$$\epsilon_s(p) := \sup\{\, n \in \mathbf{N} : p^n \mid s!(m) \text{ for some } m \in \mathbf{Z}_+ \,\}.$$

Now define

$$\mathbf{Z}(s) = \Big\{\, \frac{m}{s!(n)} : m \in \mathbf{Z} \text{ and } n \in \mathbf{Z}_+ \,\Big\}. \tag{7.8}$$

Then $\mathbf{Z}(s)$ is an additive subgroup of $\mathbf{Q}$ which is ordered with respect to the order inherited from $\mathbf{Q}$. Since it follows that

$$\mathbf{Z}(s) = \Big\{\, \frac{m}{p_1^{n_1}\cdots p_k^{n_k}} : m \in \mathbf{Z},\ p_j \text{ is prime, and } p_j^{n_j} \mid s!(n) \text{ for some } n \in \mathbf{Z}_+ \,\Big\}, \tag{7.9}$$

it is not hard to see that $\mathbf{Z}(s) = \mathbf{Z}(s')$ if and only if $\epsilon_s = \epsilon_{s'}$ (as functions on the set $\mathbf{P}$ of primes).

Now recall that if $s : \mathbf{Z}_+ \to \mathbf{Z}_+$ is a function as above, then we write M_s for the associated UHF algebra which is the inductive limit of the sequence

$$M_{s!(1)} \xrightarrow{\varphi_1} M_{s!(2)} \xrightarrow{\varphi_2} M_{s!(3)} \xrightarrow{\varphi_3} \dots$$

of matrix algebras and canonical unital maps $\varphi_n : M_{s!(n)} \to M_{s!(n+1)}$.

Proposition 7.10. *There is an order isomorphism $\rho : K_0(M_s) \to \mathbf{Z}(s)$ such that $\rho\big([\mathbf{1}]_{M_s}\big) = 1$.*

Proof. Let $A_n = M_{s!(n)}$. Then $K_0(A_n)^+ = \mathbf{N}[e_{11}^n]$. Let $\rho^n : K_0(A_n) \to \mathbf{Z}(s)$ be the unique positive homomorphism such that

$$\rho^n\big([e_{11}^n]_{A_n}\big) = \frac{1}{s!(n)}.$$

Since

$$(\varphi_n)_*\big([e_{11}^n]\big) = s(n+1) \cdot [e_{11}^{n+1}]_{A_{n+1}},$$

the diagram

$$\begin{tikzcd} K_0(A_n) \arrow[rr, "(\varphi_n)_*"] \arrow[dr, "\rho^n"'] & & K_0(A_{n+1}) \arrow[dl, "\rho^{n+1}"] \\ & \mathbf{Z}(s) & \end{tikzcd}$$

commutes for all n. Since Theorem 7.7 implies that $K_0(M_s)$ is the direct limit of the $K_0(A_n)$, there is a unique homomorphism $\rho : K_0(M_s) \to \mathbf{Z}(s)$ such that

$$\begin{tikzcd} K_0(M_s) \arrow[rr, "\rho"] & & \mathbf{Z}(s) \\ & K_0(A_n) \arrow[ul, "\varphi_*^n"] \arrow[ur, "\rho^n"'] & \end{tikzcd}$$

commutes for all n.

Notice that ρ is clearly surjective; in fact,

$$\rho\big(K_0(M_s)^+\big) = \mathbf{Z}(s)^+. \tag{7.10}$$

Furthermore, if $\rho\big(\varphi_*^n\big(m \cdot [e_{11}^n]\big)\big) = 0$, then $m\rho^n\big([e_{11}^n]\big) = \frac{m}{s!(n)} = 0$, and $m = 0$. It follows that ρ is an isomorphism and an order isomorphism in view of (7.10).

To complete the proof we check that

$$\rho\big([\mathbf{1}]_{M_s}\big) = \rho^n\big([\mathbf{1}]_{A_n}\big) = \rho^n\big(s!(n) \cdot [e_{11}^n]\big) = 1. \qquad \square$$

Theorem 7.11. *Two Glimm algebras M_s and $M_{s'}$ are isomorphic if and only if $\epsilon_s = \epsilon_{s'}$ as functions on $\mathbf{P}$.*

Proof. By Theorem 5.29 on page 97, $M_s \cong M_{s'}$ implies that $\epsilon_s = \epsilon_{s'}$. But if $\epsilon_s = \epsilon_{s'}$, then $\mathbf{Z}(s) = \mathbf{Z}(s')$. Then it follows from Proposition 7.10 that there is a unital order isomorphism $\tau : K_0(M_s) \to K_0(M_{s'})$. Therefore $M_s \cong M_{s'}$ by Elliot's Theorem 7.1 on page 117. $\square$

Let's close with a fascinating application of Elliot's Theorem.

Example 7.12. Let $A = \varinjlim (A_n, \varphi_n)$ be the AF-algebra where

$$A_n := M_{2^{n-1}} \oplus M_{2^{n-1}}$$

and $\varphi_n : A_n \to A_{n+1}$ is given by

$$\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \mapsto \begin{pmatrix} \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \end{pmatrix}.$$

Thus A has Bratteli diagram:

$$\begin{tikzcd}
1 \arrow[d] \arrow[dr] & 1 \arrow[d] \arrow[dl] \\
2 \arrow[d] \arrow[dr] & 2 \arrow[d] \arrow[dl] \\
4 \arrow[d, dotted, no head] \arrow[dr, dotted, no head] & 4 \arrow[d, dotted, no head] \arrow[dl, dotted, no head] \\
{} & {}
\end{tikzcd}.$$

It is an exercise to show that $K_0(A)$ is order isomorphic to the *dyadic rationals*:

$$\mathbf{D} := \Big\{ \frac{m}{2^n} : m \in \mathbf{Z} \text{ and } n \in \mathbf{N} \Big\}. \tag{7.11}$$

Then, using Elliot's Theorem 7.1 and Proposition 7.10, we can conclude—I think remarkably—that A is isomorphic to the UHF algebra $M(2^\infty) := M_s$, where $s : \mathbf{Z}_+ \to \mathbf{Z}_+$ is the constant function $s(n) = 2$. (Note that it is not even obvious at first blush that A is even simple.)

Exercises

E 7.2.1. Show that the equivalence relation defined on W on page 123 is given by $(x, n) \sim (y, m)$ if there is a $k \geq \max\{n, m\}$ such that $\varphi_{nk}(x) = \varphi_{mk}(y)$.

—Answer on page 141

E 7.2.2. Establish the following assertions made in this section.

(a) The set $\mathbf{Z}(s)$ as defined in (7.8) is an additive subgroup of $\mathbf{Q}$.

(b) $\mathbf{Z}(s)$ coincides with the set described in (7.9).

(c) We have $\mathbf{Z}(s) = \mathbf{Z}(s')$ if and only if $\epsilon_s = \epsilon_{s'}$ as functions on $\mathbf{P}$.

E 7.2.3. Suppose that $\big(G, \alpha^n\big) = \varinjlim\big(G_n, \alpha_n\big)$ and $\big(H, \beta^n\big) = \varinjlim\big(H_n, \beta_n\big)$ are direct limits of ordered (abelian) groups as in Example 7.5. Suppose also that we have positive homomorphisms $\tau_n : H_n \to G_n$ such that

$$\begin{tikzcd} G_n \arrow[r, "\alpha_n"] & G_{n+1} \\ H_n \arrow[u, "\tau_n"] \arrow[r, "\beta_n"'] & H_{n+1} \arrow[u, "\tau_{n+1}"'] \end{tikzcd}$$

commutes for all n. Then there is a unique positive homomorphism $\psi : H \to G$ such that $\psi\big(\beta^n(h)\big) = \alpha^n\big(\tau_n(h)\big)$.

—Answer on page 141

E 7.2.4. Let A be the AF-algebra from Example 7.12.

(a) Use Theorem 7.7 to show that $K_0(A)$ is order isomorphic to the inductive limit

$$\mathbf{Z}^2 \xrightarrow{L_M} \mathbf{Z}^2 \xrightarrow{L_M} \mathbf{Z}^2 \xrightarrow{L_M} \cdots, \tag{7.12}$$

where L_M is the map given by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

(b) Show that $K_0(A)$ is order isomorphic to the dyadic rationals $\mathbf{D}$ (as in (7.11)).

(c) Use Elliot's Theorem to conclude that $A \cong M(2^\infty)$ as asserted in Example 7.12.

—Answer on page 141

Answers to Some of the Exercises

E 1.1.8. Fix $x_0 \in X \setminus U$. By definition of E, there is a $f_{x_0} \in J$ with $f_{x_0}(x_0) \neq 0$. Since $|f|^2 = \bar{f}f \in J$ if $f \in J$, we may as well assume that $f_{x_0}(x) \geq 0$ for all $x \in X$, and since J is a subalgebra, we may also assume that $f_{x_0}(x_0) > 1$. Since $X \setminus U$ is compact, there are $x_1, \dots x_n \in X$ so that $f = \sum_k f_{x_k}$ satisfies $f \in J$ and $f(x) > 1$ for all $x \in X \setminus U$. Observe that $g = \min(1, 1/f)$ is in $C(X)$[3]. Since $fg \in J$, we are done with part (a).

Notice that we have proved that there is a $f \in J$ such that $0 \leq f(x) \leq 1$ for all $x \in X$ and $f(x) = 1$ for all $x \notin U$. Thus if h is any function in $I(E)$ and $\epsilon > 0$, then $U = \{\, x \in X : |h(x)| < \epsilon \,\}$ is a neighborhood of E in X. Then we can choose $f \in J$ as above and $\|fh - h\|_\infty < \epsilon$. Thus $h \in \overline{J} = J$. This suffices as we have $J \subseteq I(E)$ by definition. (Notice that if $E = \emptyset$, then we can take $U = \emptyset$ in the above and then $J = C(X)$.)

Remark: Notice that we have established a 1-1 correspondence between the closed subsets E of X and the closed ideals J of $C(X)$: it follows immediately from Urysohn's Lemma[4] that if E is closed and $x \notin E$, then there is a $f \in I(E)$ with $f(x) \neq 0$. Thus $I(E) \neq I(F)$ if E and F are distinct closed sets.

E 1.1.9. Suppose $\tilde{f}$ is continuous at $x = \infty$, and that $\epsilon > 0$. Then $U = \{\, \tilde{x} \in X^+ : |\tilde{f}(\tilde{x})| < \epsilon \,\}$ is an open neighborhood of ∞ in X^+. But then $X \setminus U$ is compact; but that means $\{\, x \in X : |f(x)| \geq \epsilon \,\}$ is compact. That is, $f \in C_0(X)$ as required.

For the converse, suppose that $f \in C_0(X)$, and that V is open in $\mathbf{C}$. If $0 \notin V$, then $\tilde{f}^{-1}(V) = f^{-1}(V)$ is open in X, and therefore, open in X^+. On the other hand, if $0 \in V$, then there is a $\epsilon > 0$ so that $\{\, z \in \mathbf{C} : |z| < \epsilon \,\} \subseteq V$. Thus, $X^+ \setminus \tilde{f}^{-1}(V) = \{\, x \in X : f(x) \notin V \,\} \bigcap \{\, x \in X : |f(x)| \geq \epsilon \,\}$. Since the first set is

[3] If $a, b \in C(X)$, then so are $\min(a, b) = (a+b)/2 - |a-b|/2$ and $\max(a, b) = (a+b)/2 + |a-b|/2$. In the above, we can replace f by $\max(f, 1/2)$ without altering g.

[4] For a reference, see Pedersen's *Analysis Now*: Theorems 1.5.6 and 1.6.6 or, more generally, Proposition 1.7.5.

closed and the second compact, $X^+ \setminus \tilde{f}^{-1}(V)$ is a compact subset of X, and $\tilde{f}^{-1}(V)$ is a open neighborhood of ∞ in X^+. This proves part (a).

Part (b) is immediate: each $f \in C_0(X)$ has a (unique) extension to a function in $C(X^+)$ and this identifies $C_0(X)$ with the ideal $I\big(\{\,\infty\,\}\big)$ in $C(X^+)$. In view of question 1.1.8 above, $I\big(\{\,\infty\,\}\big)$ is maximal among closed ideals in $C(X^+)$, and, as maximal ideals are automatically closed, maximal among all proper ideals.

E 1.1.10. Suppose that J is a closed ideal in $C_0(X)$. Then J is, in view of question 1.1.9(b) above, a closed subalgebra of $C(X^+)$. I claim the result will follow once it is observed that J is actually an ideal in $C(X^+)$. In that case, $J = I(E \cup \{\,\infty\,\})$, where $E \subseteq X$ is such that $E \cup \{\,\infty\,\}$ is closed in X^+. Thus $X^+ \setminus (E \cup \{\,\infty\,\}) = X \setminus E$ is open in X, and E is closed in X.

The easy way to verify the claim, is to observe that, in view of the fact that $C_0(X)$ is a maximal ideal in $C(X^+)$, $C(X^+) = \{\, f + \lambda : f \in C_0(X) \text{ and } \lambda \in \mathbf{C} \,\}$. (Here $\lambda \in \mathbf{C}$ is identified with the constant function on X^+.) Then, since J is an algebra, $f(g + \lambda) = fg + \lambda f$ belongs to J whenever f does.

E 1.1.11. If there is a $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in A$, then consider the one-point compactification of $X \setminus \{x_0\}$. Otherwise, consider the one-point compactification of X.

E 1.2.1. Using the "hint", suppose that $a \in \operatorname{Inv}(A)$ and let $b = a - h$ for some $h \in A$ with

$$\|h\| \leq \frac{\|a^{-1}\|^{-1}}{2}. \tag{7.13}$$

Then standard arguments show that $b \in \operatorname{Inv}(A)$. (Recall that if $\|c\| < 1$, then $1 - c \in \operatorname{Inv}(A)$ and $(1 - c)^{-1} = 1 + c + c^2 + \cdots$.) Now we observe that

$$\begin{aligned}
\|b^{-1} - a^{-1}\| &= \|(a-h)^{-1} - a^{-1}\| \\
&= \big\|\big((1 - a^{-1}h) - 1\big)a^{-1}\big\| \\
&\leq \Big\|\sum_{n=1}^{\infty}(a^{-1}h)^n\Big\|\|a^{-1}\| \\
&\leq \sum_{n=1}^{\infty}\|a^{-1}h\|^n\|a^{-1}\| \\
&= \frac{\|a^{-1}h\|\|a^{-1}\|}{1 - \|a^{-1}h\|}
\end{aligned}$$

which, in view of the above and $\|a^{-1}h\| \le \|a^{-1}\|\|h\|$, is

$$\le 2\|a^{-1}\|^2\|h\|.$$

Since the latter term tends to 0 with h, we're done.

However, Peng Peng Yu came up with a cleaner argument that does not require A to be a Banach algebra. Here it is enough that A be a normed algebra (and therefore with a submultiplicative norm).

Fix $a \in A$ and $\epsilon > 0$. Let $\delta = \frac{1}{2}\min\bigl(\epsilon\|a^{-1}\|^{-2}, \|a^{-1}\|^{-1}\bigr)$. We just need to verify that if $b \in B_\delta(a) \cap \operatorname{Inv}(A)$, then $\|a^{-1} - b^{-1}\| < \epsilon$. However, we first notice that by the "reverse triangle inequality",

$$\begin{aligned}\|b^{-1}\| - \|a^{-1}\| &\le \|b^{-1} - a^{-1}\| \\ &= \|b^{-1}(a-b)a^{-1}\| \\ &\le \|b^{-1}\|\|a-b\|\|a^{-1}\|\end{aligned}$$

which, since $\|a-b\| < \frac{1}{2}\|a^{-1}\|^{-1}$, is

$$< \frac{1}{2}\|b^{-1}\|.$$

In particular, this implies that

$$\|b^{-1}\| < 2\|a^{-1}\| \quad \text{if } b \in B_\delta(a) \cap \operatorname{Inv}(A). \tag{7.14}$$

But then if $b \in B_\delta(a) \cap \operatorname{Inv}(A)$, we also have $\|a-b\| < \frac{\epsilon}{2}\|a^{-1}\|^{-2}$. Therefore

$$\begin{aligned}\|b^{-1} - a^{-1}\| &= \|b^{-1}(a-b)a^{-1}\| \\ &\le \|b^{-1}\|\|a-b\|\|a^{-1}\| \\ &< \epsilon.\end{aligned}$$

This is what we wanted to show.

E 1.3.4. First compute that[5]

$$\begin{aligned}\|f*g\|_1 &= \int_0^1 |f*g(t)|\,dt \\ &\le \int_0^1 \int_0^t |f(t-s)g(s)|\,ds\,dt\end{aligned}$$

which, using Tonelli's Theorem, is

$$\begin{aligned}&= \int_0^1 |g(s)|\Bigl(\int_s^1 |f(t-s)|\,dt\Bigr)\,ds \\ &= \int_0^1 |g(s)|\Bigl(\int_0^{1-s} |f(u)|\,du\Bigr)\,ds \\ &\le \|f\|_1\|g\|_1.\end{aligned}$$

To show that $f*g = g*f$ it suffices, in view of the above, to consider continuous functions. Thus, the usual calculus techniques apply. In particular,

$$\begin{aligned}f*g(t) &= \int_0^t f(t-s)g(s)\,ds \\ &= -\int_t^0 f(u)g(t-u)\,du = g*f(t).\end{aligned}$$

This proves (a). However, (b) is a simple induction argument.

Now for (c): the calculation (1.11) shows that $\mathrm{alg}(f_0)$ contains all polynomials. Since the polynomials are uniformly dense in $C[0,1]$, and the later is dense in L^1, we can conclude that $\mathrm{alg}(f_0)$ is norm dense.

Next, observe that (1.12) not only implies that $\rho(f_0) = 0$, but that $\rho(f_0^k) = 0$ as well for any positive integer k. However, it is not immediately clear that every element of $\mathrm{alg}(f_0)$ has spectral radius zero. However, there is an easy way to see this. Let $\widetilde{A}$ be the unitalization of A (i.e., $\widetilde{A} := A \oplus \mathbf{C}$), and recall that $a \in A$ has spectral

[5]For a reference for Tonelli's Theorem (the 'uselful' version of Fubini's Theorem), see [*Analysis Now*, Corollary 6.6.8], or much better, see Royden's *Real Analysis*. On the other hand, if you are worried about the calculus style manipulation of limits, consider the integrand

$$F(s,t) = \begin{cases} |f(t-s)g(s)| & \text{if } 0 \le s \le t \le 1, \text{ and} \\ 0 & \text{otherwise.}\end{cases}$$

radius zero (a is called *quasi-nilpotent*) if and only if $\tilde{h}(a) = 0$ for all $\tilde{h} \in \widetilde{\Delta} = \Delta(\widetilde{A})$. Since each $\tilde{h}$ is a continuous algebra homomorphism, $\ker(\tilde{h})$ is a closed ideal in $\widetilde{A}$, and it follows that the collection of quasi-nilpotent elements is actually a *closed ideal* of A given by[6]

$$\operatorname{rad}(A) = \bigcap\nolimits_{\tilde{h} \in \widetilde{\Delta}} \ker(\tilde{h}).$$

Since each f_0^k is in $\operatorname{rad}(A)$, so is the *closed* algebra (in fact, the closed ideal) generated by f_0. Thus, $\operatorname{rad}(A) = A$ in this case, which is what was to be shown.

Of course, (d) is an immediate consequence of (c): if $\rho \in \Delta(A)$, then by defintion there is a $f \in A$ such that $\rho(f) \neq 0$. But then $\rho(f) \geq |h(f)| > 0$, which contradicts the fact that $\operatorname{rad}(A) = A$.

E 1.3.6. By Theorem J from lecture, $\sigma(f) = \{ \varphi(z) : z \in \mathbf{T} \}$. Since φ never vanishes, $0 \notin \sigma(f)$ and f is invertible. Let $g = f^{-1}$, and consider $\hat{g}$. Since $g * f = 1_A$, we have $\hat{g}\hat{f} = \hat{g}\varphi = 1_{C(\mathbf{T})}$. It follows that $\hat{g} = \psi$, and hence ψ has an absolutely convergent Fourier series — namely g.

E 2.1.1. In part (b), just take $y = \|x\|^{-1}x^*$.

E 2.1.2. Since A is a Banach algebra, $\|x\|^2 \leq \|x^*x\| \leq \|x^*\|\,\|x\|$, which implies that $\|x\| \leq \|x^*\|$. Replacing x by x^*, we get $\|x^*\| \leq \|x^{**}\| = \|x\|$. Thus, A is a Banach $*$-algebra, and the C^*-norm equality follows from the first calculation and that fact that in any Banach $*$-algebra, $\|x^*x\| \leq \|x\|^2$.

E 2.1.3. The real issue is to see that the direct sum is complete. So suppose that $\{ a_n \}$ is Cauchy in $\bigoplus_{i \in I} A_i$. Then, clearly, each $\{ a_n(i) \}$ is Cauchy in A_i, and hence there is $a(i) \in A_i$ such that $a_n(i) \to a(i)$. If $\epsilon > 0$, choose N so that $n, m \geq N$ imply that $\|a_n - a_m\| < \epsilon/3$. I claim that if $n \geq N$, then $\|a_n - a\| < \epsilon$. This will do the trick.

But for each $i \in N$, there is a $N(i)$ such that $n \geq N(i)$ implies that $\|a_n(i) - a(i)\| < \epsilon/3$. Then if $n \geq N$, we have

$$\|a_n(i) - a(i)\| \leq \|a_n(i) - a_{N(i)}(i)\| + \|a_{N(i)}(i) - a(i)\| < \frac{2\epsilon}{3}.$$

But then $n \geq N$ implies that

$$\sup_{i \in I} \|a_n(i) - a(i)\| \leq \frac{2\epsilon}{3} < \epsilon$$

[6]This result is of interest in its own right. Note that A is always a maximal ideal in $\widetilde{A}$, and so $\operatorname{rad}(A)$ is always contained in A itself.

as required.

E 2.1.4. If $1 \in A$, then it is easy to provide an inverse to the given map.

The interesting bit is when A is non-unital to begin with. Since A is complete, $\mathscr{B}(A)$ is a Banach algebra with respect to the operator norm. The set $B = \{\, lI + L_x : l \in \mathbf{C}, x \in A \,\}$ is clearly a subalgebra which admits an involution: $(lI + L_x)^* = \bar{l}I + L_{x^*}$. Notice that we have

$$\|L_x\| = \sup_{\|y\|=1} \|xy\| = \|x\|$$

(problem 1(b) above). Since $L_{lx} = lL_x$, $L_{(x+y)} = L_x + L_y$, $L_{xy} = L_x \circ L_y$, and $L_{x^*} = L_x^*$, the map $x \mapsto L_x$ is an isometric $*$-isomorphism of A onto $B_0 = \{\, L_x \in \mathscr{B}(A) : x \in A \,\}$. It follows that B_0 is complete and therefore closed in $\mathscr{B}(A)$. Therefore, since $I \notin B_0$ (because $e \notin A$) and since the invertible elements in $\mathscr{B}(A)$ are open, there is a $\delta > 0$ such that $\|I - L_x\| \geq \delta$ for all $x \in A$. So to see that B is also closed, suppose that $l_n I + L_{x_n} \to L$ in $\mathscr{B}(A)$. Passing to a subsequence and relabeling, we may assume that $l_n \neq 0$ for all n. (If infinitely many l_n are zero, then $L \in B_0$.) Thus, $|l_n|\|I + l_n^{-1} L_{x_n}\| \to \|L\|$. Since $\|I + l_n^{-1} L_{x_n}\| \geq \|\delta\|$, it follows that $\{\, l_n \,\}$ must be bounded, and hence must have a convergent subsequence. Therefore $L \in B$, and B is a Banach algebra.

Finally,

$$\begin{aligned}
\|lI + L_x\|^2 &= \sup_{\|y\|=1} \|ly + xy\|^2 = \sup_{\|y\|=1} \|(ly + xy)^*((ly + xy)\| \\
&= \sup_{\|y\|=1} \|y^*(\bar{l}I + L_{x^*})\big((lI + L_x)(y)\big)\| \\
&\leq \sup_{\|y\|=1} \|(lI + L_x)^*\big((lI + L_x)(y)\big)\| \\
&= \|(lI + L_x)^*(lI + L_x)\|.
\end{aligned}$$

It now follows from problem 2.1.2 that B is a C*-algebra. It is immediate that $(x, \lambda) \mapsto \lambda I + L_x$ is an (algebraic) isomorphism of A^1 onto B (note that you need to use the fact that A in non-unital to see that this map is injective). Of course, $\|(x, \lambda)\| := \|\lambda I + L_x\|_B$ is the required norm on A^1.

E 2.2.4. Suppose that $f : \mathbf{R} \to \mathbf{C}$ is continuous and that $x_n \to x$ in $A_{\text{s.a.}}$. We need to see that $f(x_n) \to f(x)$ in A. Since we may write $f = f_1 + if_2$ with f_i real-valued and since $f(x_n) = f_1(x_n) + if_2(x_n)$, we may as well assume that f itself is real-valued. Furthermore, since addition and multiplication are norm-continuous in A, we certainly have $p(x_n) \to p(x)$ for any polynomial; this is proved in the same was as one proves that any polynomial is continuous in calculus. Clearly there is a constant $M \in \mathbf{R}^+$ so that $\|x_n\| \le M$ for all n. Thus $\rho(x_n) \le M$ and $\sigma(x_n) \subseteq [-M, M]$ for all n. Similarly, $\sigma(x) \subseteq [-M, M]$ as well. By the Weierstrass approximation theorem, given $\epsilon > 0$, there is a polynomial p such that $|f(t) - p(t)| < \epsilon/3$ for all $t \in [-M, M]$. Thus for each n,

$$\|f(x_n) - p(x_n)\| = \sup_{t \in \sigma(x_n)} |f(t) - p(t)| < \epsilon/3. \tag{$\dagger$}$$

(Notice that $f(x_n)$ is the image of $f|_{\sigma(x_n)}$ by the isometric $*$-isomorphism of $C\big(\sigma(x_n)\big)$ onto the abelian C*-subalgebra of A generated by e and x_n. Then (†) follows because $f(x_n) - p(x_n)$ is the image of $(f - p)|_{\sigma(x_n)}$ which has norm less than $\epsilon/3$ in $C\big(\sigma(x_n)\big)$ since $\sigma(x_n) \subseteq [-M, M]$.) Of course, (†) holds with x_n replaced by x as well. Now choose N so that $n \ge N$ implies that $\|p(x_n) - p(x)\| < \epsilon/3$. Therefore for all $n \ge N$,

$$\|f(x_n) - f(x)\| \le \|f(x_n) - p(x_n)\| + \|p(x_n) - p(x)\| + \|p(x) - f(x)\| < \epsilon.$$

The conclusion follows.

E 2.2.5. The spectrum of any element of $C(X)$ is simply its range, so we immediately have $\sigma_{C(\mathbf{T})}(f) = \mathbf{T}$. But $\lambda - f$ is invertible in $A(D)$ only when $(\lambda - f)^{-1}$ has an analytic extension to D, but if $\lambda \in D$, then this is impossible since

$$\int_{|z|=1} \frac{1}{\lambda - z}\, dz = 2\pi i \quad \text{if } \lambda \in D.$$

On the other hand, if $|\lambda| > 1$, then $\lambda - f$ is clearly in $G\big(A(D)\big)$. Therefore $\sigma_{A(D)}(f) = \overline{D}$ as claimed.

E 2.3.4. That (b) implies (c) is easy. To see that (c) implies (b), note that $(UU^*U - U)(UU^*U - U)^* = (UU^*)^3 - 2(UU^*)^2 + UU^*$, which is zero. But in a C^*-algebra, $x^*x = 0$ implies that $x = 0$. Therefore $UU^*U - U = 0$.

Now replacing U by U^* gives us the fact that (b), (c), and (d) are equivalent.

But if U^*U is a projection, then the range of U^*U is exactly $\ker(U^*U)^\perp$. I claim $\ker(T^*T) = \ker(T)$ for any bounded operator. Obviously, $\ker(T) \subseteq \ker(T^*T)$. On the other hand, if $T^*T(x) = 0$, then $\langle T^*Tx, x\rangle = 0 = \langle Tx, Tx\rangle = |Tx|^2$. This proves the claim.

It follows from the previous paragraph that if $x \in \ker(U)^\perp$, then $U^*Ux = x$. But then $|Ux|^2 = \langle Ux, Ux\rangle = \langle U^*Ux, x\rangle = \langle x, x\rangle = |x|^2$. Thus, (d) implies (a).

Finally, if (a) holds, then the polarization identity implies that $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all $x, y \in \ker(U)^\perp$. Now suppose $x \in \ker(U)^\perp$. On the one hand, $z \in \ker(U)^\perp$

implies that $\langle U^*Ux, z\rangle = \langle Ux, Uz\rangle = \langle x, z\rangle$. While on the other hand, $z \in \ker(U)$ implies that $\langle U^*Ux, z\rangle = \langle Ux, Uz\rangle = 0 = \langle x, z\rangle$. We have shown that $\langle U^*Ux, y\rangle = \langle x, y\rangle$ for all $y \in \mathcal{H}$ and $x \in \ker(U)^\perp$; therefore the restriction of U^*U to $\ker(U)^\perp$ is the identity. But U^*U is certainly zero on $\ker(U)$. In other words, U^*U is the projection onto $\ker(U)^\perp$, and (a) implies (d).

Of course we just proved above that if U is partial isometry, then U^*U is the projection onto $\ker(U)^\perp$. I'm glad everyone (eventually anyway) realized this is what I meant. Sorry if you wasted time here. Of course, taking adjoints in part (b) shows that U^* is a partial isometry, so $UU^* = U^{**}U^*$ is the projection onto $\ker(U^*)^\perp$. It is standard nonsense that, for any bounded operator T, $\ker(T^*) = T(\mathcal{H})^\perp$ (see, for example, *Analysis Now*, 3.2.5). Thus, UU^* is the projection onto $\ker(U^*)^\perp$, which is the closure of the range of U. However, the range of U is the isometric image of the closed, hence complete, subspace $\ker(U)^\perp$. Thus the range of U is complete, and therefore, closed. Thus, UU^* is the projection onto the range of U as claimed.

E 2.4.1. Clearly $IJ \subseteq I \bigcap J$. Suppose $a \in I \bigcap J$, and that $\{ e_\alpha \}_{\alpha \in A}$ is an approximate identity for J. Then ae_α converges to a in J. On the other hand, for each α, $ae_\alpha \in IJ$. Thus, $a \in IJ$. This proves part (a).

For part (b), consider $a \in A$ and $b \in I$. Again let $\{ e_\alpha \}_{\alpha \in A}$ be an approximate identity for J. Then $ab = \lim_\alpha a(e_\alpha b) = \lim_\alpha (ae_\alpha)b$, and the latter is in I, since I is closed and $ae_\alpha \in J$ for all α. This suffices as everything in sight is $*$-closed, so I must be a two-sided ideal in A.

E 2.4.2. Let $\{ e_\lambda \}_{\lambda \in \Lambda}$ be the net constructed in the proof of the Theorem. If $D = \{ x_k \}_{k=1}^\infty$ is dense in $A_{\text{s.a.}}$, then define $e_n = e_{\lambda_n}$ where $\lambda_n = \{ x_1, \ldots, x_n \}$. Since properties (1)–(3) are clear, we only need to show that $xe_n \to x$ for all $x \in A$. (This will suffice by taking adjoints.) As we saw in the proof of the Theorem, $\|xe_n - x\|^2 = \|x^*x - x^*xe_n\|$, so we may as well assume that $x \in A_{\text{s.a.}}$. But then if $x \in \{ z_1, \ldots, z_n \} = \lambda$, we have $\|x - xe_\lambda\|^2 \le 1/4n$.

So fix $x \in A_{\text{s.a.}}$ and $\epsilon > 0$. Choose $y \in D$ such that $\|x - y\| < \epsilon/3$. Finally, choose N so that $y \in \{ x_1, \ldots, x_N \} = \lambda_N$, and such that $1/4N < \epsilon/3$. Then, since $\|e_n\| \le 1$, $n \ge N$ implies that

$$\|x - xe_n\| \le \|x - y\| + \|y - ye_n\| + \|ye_n - xe_n\| < \epsilon.$$

This suffices.

E 3.1.3. Let $h \in \mathcal{H}$ and $\epsilon > 0$ be given. By assumption there are vectors $h_1, \ldots, h_n \in \mathcal{H}$ and elements $x_1, \ldots, x_n \in A$ such that $\|h - \sum_{i=1}^n \pi(x_i)h_i\| < \epsilon$. On the other hand, $e_\lambda x_i \to x_i$ for each i. Thus $\pi(e_\lambda x_i) \to \pi(x_i)$ in norm. Choose $\lambda_0 \in \Lambda$ so that $\lambda \geq \lambda_0$ implies that $\| \sum_{i=1}^n \pi(x_i)h_i - \sum_{i=1}^n \pi(e_\lambda x_i)h_i\| < \epsilon$. Then

$$\begin{aligned}\|h - \pi(e_\lambda)h\| \leq \Bigl\|h - \sum_{i=1}^n \pi(x_i)h_i\Bigr\| + \\ \Bigl\|\sum_{i=1}^n \pi(x_i)h_i - \sum_{i=1}^n \pi(e_\lambda x_i)h_i\Bigr\| + \Bigl\|\pi(e_\lambda)\Bigl(\sum_{i=1}^n \pi(x_i)h_i - h\Bigr)\Bigr\| < 3\epsilon.\end{aligned}$$

E 3.3.3. We must have $\{\, h \in \mathcal{H} : \|Th\| < 1 \,\}$ is open in the weak topology. Hence there are vectors $k_1, \dots, k_n \in \mathcal{H}$ such that $\sum_{i=1}^n |(h \mid k_i)| < \epsilon$ implies $\|T(h)\| < 1$. Let $V := \operatorname{span}\{k_i\}$. If $T(\mathcal{H})$ is infinite dimensional, then $T(V^\perp) \neq \{0\}$. Suppose $h \in V^\perp$ and $Th \neq 0$. Then we must have $\|T(\lambda h)\| = |\lambda|\|Th\|$ for all λ. This is clearly impossible.

E 3.5.1. By assumption, $\mathrm{id} : \pi(A) \to B(\mathcal{H})$ is irreducibe. If $\pi(A) \bigcap \mathcal{K}(\mathcal{H}) \neq \{\, 0 \,\}$, then the restriction of id to the *ideal* $\pi(A) \bigcap \mathcal{K}(\mathcal{H})$ is non-zero, and hence irreducible. But then $\pi(A) \bigcap \mathcal{K}(\mathcal{H})$ is an irreducible C^*-subalgebra of $\mathcal{K}(\mathcal{H})$, and is therefore all of $\mathcal{K}(\mathcal{H})$.

E 3.5.3. See Remark 8.12 in my book on crossed products.

E 3.5.5. Look at Example A.31 in my book with Iain Raeburn on Morita equivalence.

E 3.5.6. See example A.32 in my book on Morita equivalence (written with Iain Raeburn). (a) If T is bounded from below, then $T\mathcal{H}$ is complete and therefore closed. Furthermore, $\ker T = \{\, 0 \,\}$. If $T = T^*$, then $T\mathcal{H}^\perp = T^*\mathcal{H}^\perp = \ker T = \{\, 0 \,\}$. Therefore T is a bounded bijection from $\mathcal{H}$ onto $\mathcal{H}$, and T^{-1} is bounded by the Closed Graph Theorem.

(b) Let $J_\epsilon = \{\, f \in C\bigl(\sigma(T)\bigr) : f(\lambda) = 0 \text{ if } |\lambda| \leq \epsilon \,\}$. Notice that if $f \in J_\epsilon$ and $g(\lambda) = \lambda$ for all $\lambda \in \sigma(T)$, then $g^2|f|^2 \geq \epsilon^2|f|^2$. It follows that $T^2 f(T)^* f(T) \geq \epsilon^2 f(T)^* f(T)$. Thus,

$$\begin{aligned}|Tf(T)\xi|^2 &= \langle T^2 f(T)^* f(T)\xi, \xi\rangle \\ &\geq \epsilon^2 \langle f(T)^* f(T)\xi, \xi\rangle \\ &= \epsilon^2 |f(T)\xi|^2.\end{aligned}$$

Now let $\xi \in M_\epsilon$. Let $\{\, f_\lambda \,\}$ be an approximate identity in J_ϵ. Then we see that $f_\lambda(T)\xi \to \xi$. (Approximate ξ by $\sum_{i=1}^n g_i(T)\xi_i$ with $g_i \in J_\epsilon$ and $\xi_i \in \mathcal{H}$.) Thus

$$|T\xi|^2 = \lim_\lambda |Tf_\lambda(T)\xi| \geq \epsilon^2 \lim_\lambda |f_\lambda(T)\xi|^2 = \epsilon^2|\xi|^2.$$

This proves that T is bounded below on M_ϵ. But since we have $TM_\epsilon \subseteq M_\epsilon$ by construction, we have $TM_\epsilon = M_\epsilon$ by part (a).

(c) Let P_n be the projection onto $M_{\frac{1}{n}}$. Define

$$f_n(\lambda) = \begin{cases} 0 & \text{if } 0 \le \lambda \le \frac{1}{n}, \\ 2(\lambda - \frac{1}{n}) & \text{if } \frac{1}{n} \le \lambda < \frac{2}{n}, \\ \lambda & \text{if } \lambda \ge \frac{2}{n}. \end{cases}$$

Then $f_n \in J_{\frac{1}{n}}$ and $f_n \to g$ uniformly on $\sigma(T)$, where $g(\lambda) = \lambda$ for all $\lambda \in \sigma(T)$. Thus $f_n(T) \to T$ and $P_n f_n(T) = f_n(T)$. If each $M_{\frac{1}{n}}$ were finite dimensional, then P_n, and hence $f_n(T)$, would be finite rank. Then T would be compact.

So choose ϵ so that $\dim M_\epsilon = \aleph_0 = \dim \mathcal{H}$. Then there is a partial isometry $V : \mathcal{H} \to \mathcal{H}$ such that $V\mathcal{H} = M_\epsilon$. Then V^*TV is bounded below on $\mathcal{H}$ and has a bounded inverse by part (a).

(d) Let I be a non-zero (closed) ideal in $B(\mathcal{H})$. Since $I \cap \mathcal{K}(\mathcal{H})$ is an ideal in $\mathcal{K}(\mathcal{H})$ we must have $\mathcal{K}(\mathcal{H}) \subseteq I$ since $\mathcal{K}(\mathcal{H})$ is simple. If $I \ne \mathcal{K}(\mathcal{H})$, then I contains a non-compact operator T. Since I is a C^*-algebra, and is therefore the span of its self-adjoint elements, we may assume that T is self-adjoint. Now it follows from part (c) that I contains an invertible element, and hence that $I = B(\mathcal{H})$ as required.

(e) It follows from the previous part that the Calkin algebra $\mathcal{C}(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is simple. If $B(\mathcal{H})$ were GCR, then $CCR(\mathcal{C}(\mathcal{H})) \ne \{0\}$. Since $\mathcal{C}(\mathcal{H})$ is simple, it follows that $\mathcal{C}(\mathcal{H})$ is CCR. Thus if π is an irreducible representation of $\mathcal{C}(\mathcal{H})$ and e is the identity element of $\mathcal{C}(\mathcal{H})$, then $\pi(e) = I_{\mathcal{H}_\pi}$ is a compact operator. This forces $\mathcal{H}_\pi$ to be finite dimensional, and since the simplicity of $\mathcal{C}(\mathcal{H})$ implies that π is an $*$-isomorphism of $\mathcal{C}(\mathcal{H})$ into $\mathscr{B}(\mathcal{H}_\pi)$ (onto actually), $\mathcal{C}(\mathcal{H})$ must be finite dimensional as well. But one can easily find infinitely many orthogonal infinite dimensional projections $\{P_n\}$ in $B(\mathcal{H})$ (when $\mathcal{H}$ is infinite dimensional). The images of the P_n in $\mathcal{C}(\mathcal{H})$ are clearly independent. This contradiction completes the proof.

E 4.1.2. See [RW98, Lemma A.7].

E 4.2.3. See [Wil07, Lemma 1.89] and note that there is an extra "convex" in the statement of the result.

E 5.4.1. There is nothing to show if $\mathbf{1}_A \in A_n$. Otherwise $\operatorname{alg}(A_n \cup \mathbf{1}_A) = \{a + \lambda \mathbf{1}_A : a \in A_n \text{ and } \lambda \in \mathbf{C}\}$, which is clearly finite-dimensional if A_n is. Since finite-dimensional subspaces of normed vector spaces are closed, in this case $C^*(A_n \cup \mathbf{1}_A) = \operatorname{alg}(A_n \cup \mathbf{1}_A)$, and the later is finite-dimensional.

E 7.2.1. Just show that the relation defined in the problem is an equivalence relation, and that any equivalence relation satisfying $(x, n) \sim (\varphi_n(x), n+1)$ must contain it.

E 7.2.3. Since $H = \bigcup \beta^n(H_n)$ and $H^+ = \bigcup \beta^n(H_n^+)$, the only real issue is to see that the formula for ψ is well defined. But, if $\beta^n(h) = \beta^m(h')$, then there is a $k \geq \max\{n, m\}$ such that

$$\beta_{nk}(h) = \beta_{mk}(h').$$

Then we have

$$\begin{aligned}\alpha^n\bigl(\tau_n(h)\bigr) &= \alpha^k\bigl(\alpha_{nk}\bigl(\tau_n(h)\bigr)\bigr) = \alpha^k\bigl(\tau_k\bigl(\beta_{nk}(h)\bigr)\bigr) \\ &= \alpha^k\bigl(\tau_k\bigl(\beta_{mk}(h')\bigr)\bigr) = \alpha^m\bigl(\tau_m(h')\bigr).\end{aligned}$$

This is what we needed to see.

E 7.2.4. We retain the notations from Example 7.12.

Theorem 7.7 implies that

$$\bigl(K_0(A), \varphi_*^n\bigr) = \varinjlim\bigl(K_0\bigl(\varphi^n(A_n)\bigr), (\varphi_n)_*\bigr).$$

As in Example 6.27 on page 111, we have a commutative diagram

$$\begin{tikzcd} K_0(A_n) \arrow[r, "(\varphi_n)_*"] & K_0(A_{n+1}) \\ \mathbf{Z}^2 \arrow[u, "\tau_n", "\cong"'] \arrow[r, "L_M"] & \mathbf{Z}^2, \arrow[u, "\cong", "\tau_{n+1}"'] \end{tikzcd}$$

where the $\tau_n : \mathbf{Z}^2 \to K_0(A_n)$ are the canonical isomorphisms.

Then if we let $\bigl(H, \beta^n\bigr) = \varinjlim\bigl(\mathbf{Z}^2, L_M\bigr)$ be the inductive limit of (7.12), it follows from E 7.2.3 that there is a unique positive homomorphism $\psi : H \to K_0(A)$ such that $\psi\bigl(\beta^n(h)\bigr) = \varphi_*^*\bigl(\tau_n(h)\bigr)$. But we can also apply E 7.2.3 to the τ_n^{-1} to bet a positive homomorphism $\psi' : K_0(A) \to H$ such that $\psi'\bigl(\varphi_*^n(x)\bigr) = \beta^n(\tau_n^{-1}(x)$. Since ψ and ψ' are clearly inverses, we're done with part (a).

For part (b), we just want to see that the direct limit H, from the previous paragraph, is order isomorphic to the dyadic rationals $\mathbf{D}$. Let $\theta^n : \mathbf{Z}^2 \to \mathbf{D}$ be defined by

$$\theta^n(m, n) = \frac{m+n}{2^{n-1}}.$$

Then

$$\begin{tikzcd} \mathbf{Z}^2 \arrow[rr, "L_M"] \arrow[dr, "\theta^n"'] & & \mathbf{Z}^2 \arrow[dl, "\theta^{n+1}"] \\ & \mathbf{D}. & \end{tikzcd}$$

commutes for all n. In other words, the $\{\theta^n\}$ are compatible with the L_M. The universal property of the direct limit asserts that there is a (unique) homomorphism $\psi : H \to \mathbf{D}$ such that

$$\begin{tikzcd} H \arrow[rr, "\psi"] & & \mathbf{D} \\ & \mathbf{Z}^2 \arrow[ul, "\beta^n"] \arrow[ur, "\theta^n"'] & \end{tikzcd}$$

commutes for all n.

Since $\theta^{n+1}\big((m,0)\big) = \frac{m}{2^n}$, ψ is clearly surjective. But if $\psi\big(\beta^k\big((m,n)\big)\big) = 0$, then $\theta^n\big((m,n)\big) = 0$ and $m = -n$. But $\beta^k\big((n,-n)\big) = \beta^{k+1}\big(L_M\big((n,-n)\big)\big) = 0$. Therefore ψ is injective as well. This completes the proof of part (b).

Part (c) follows since Proposition 7.10 implies that $K_0(M_s)$ is also order isomorphic to $\mathbf{D}$.

Notation and Symbol Index

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