QUADRATIC FORMS OVER LOCAL RINGS

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Abstract. These notes collect together some facts concerning quadratic forms over commutative local rings, specifically mentioning discrete valuation rings.

Let R be a commutative local ring with maximal ideal $\mathfrak{m}$, residue field k, and field of fractions K. A *symmetric bilinear form* (M, b) is a projective R-module of finite rank together with a symmetric R-module homomorphism $b : M \otimes_R M \to R$, equivalently, an element $b \in S^2(M)^\vee$. A *quadratic form* (M, q) is a projective R-module M of finite rank together with a map $q : M \to R$ satisfying the usual axioms, equivalently $q \in S^2(M^\vee)$. In particular, a quadratic form has an associated symmetric bilinear *polar form* b_q. Recall that every projective module of finite rank over a local ring is free.

We say that a symmetric bilinear form (M, b) is *nondegenerate* (resp. *regular*) if the canonical induced map $M \to M^\vee = \mathrm{Hom}_R(M, R)$ is an injection (resp. isomorphism). We say that a quadratic form is *nondegenerate* (resp. *regular*) if its associated symmetric bilinear form is. If $2 \notin R^\times$, then no regular bilinear form has odd rank. To repair this, there is a notion of *semiregularity*, due to Kneser, for quadratic forms of odd rank, see Knus [6, IV §3.1]. If $2 \in R^\times$ then semiregularity is equivalent to regularity.

Throughout, $\cong$ means isometry. If $N \subset M$ is a subset and (M, b) a bilinear form, $N^\perp = \{v \in M : b(v, N) = 0\}$ is the *orthogonal complement* (for quadratic forms this is defined via the polar bilinear form).

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1. Regular forms

Fact 1.1. *Regularity is a local property. A symmetric bilinear or quadratic form over R is regular if and only if the reduction modulo $\mathfrak{m}$ is regular.*

Fact 1.2 (Baeza [1, I Prop. 3.2], Knebusch [5, I Prop. 2])**.** *Let (M, q) be a quadratic (resp. symmetric bilinear) form over R and $N \subset M$ a R-submodule. If q_N is regular, then there is a decomposition $(M, q) \cong (N, q|_N) \perp (N^\perp, q|_{N^\perp})$.*

Fact 1.3 (Baeza [1, I Cor. 3.4])**.** *Let (M, q) be a quadratic (or symmetric bilinear) form over R. Every orthogonal decomposition $(M, q) \otimes_R k \cong (M_1', q_1') \perp (M_2', q_2')$, with (M_1', q_1') a (semi)regular quadratic or (symmetric bilinear) form over k, lifts to an orthogonal decomposition $(M, q) \cong (M_1, q_1) \perp (M_2, q_2)$, where (M_1, q_1) and (M_2, q_2) are quadratic (or symmetric bilinear) forms over R reducing to (M_1', q_1') and (M_2', q_2'), respectively, and with (M_1, q_2) is regular.*

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Proof. Let $q \otimes_R k \cong q'_1 \perp q'_2$ with q'_1 regular. Then there exists $M_1 \hookrightarrow M$ with $q|_{M_1}$ regular by Fact 1.1. But then by Fact 1.2, $q \cong q|_{M_1} \perp q_2$ for some q_2 and where $q|_{M_1} \otimes_R k \cong q'_1$. But then $q_2 \otimes_R k \cong q'_2$ by Witt cancellation for fields. □

Definition 1.4. Let $a, b \in R$. Define the standard quadratic forms $[a] = (R, x \mapsto ax^2)$ and $[a, b] = (R^2, (x, y) \mapsto ax^2 + xy + by^2)$. Then $[a]$ is semiregular if and only if $a \in R^\times$ and $[a, b]$ is regular if and only if $1 - 4ab \in R^\times$. Note: if $2 \notin R^\times$ then $[a]$ is never regular.

Let $a \in R$. Define the standard symmetric bilinear form $<a> = (R, (x, y) \mapsto axy)$. Then $<a>$ is regular if and only if $a \in R^\times$.

For every symmetric $n \times n$ matrix T over R, there is an associated symmetric bilinear form $(R^n, (v, w) \mapsto v^t T w)$.

Fact 1.5. *Every (semi)regular quadratic form (M, q) over R has a decomposition of the form*

$$(M, q) \cong \begin{cases} [a_1, b_1] \perp \ldots \perp [a_r, b_r] & \text{if the rank is even} \\ [a_1, b_1] \perp \ldots \perp [a_r, b_r] \perp [a] & \text{if the rank is odd} \end{cases}$$

for elements $a_i, b_i \in R$ such that $1 - 4a_ib_i \in R^\times$ and $a \in R^\times$. If $2 \in R^\times$, then any regular quadratic form (M, q) over R has an orthonormal basis

$$(M, q) \cong [a_1] \perp \ldots \perp [a_r]$$

for $a_i \in R^\times$.

Proof. This follows directly from the corresponding statement over the residue field k lifted to R via Fact 1.3. □

Definition 1.6. A quadratic form (M, q) is *primitive* if the ideal generated by the values $q(M)$ is the unit ideal; every regular quadratic form is primitive. A symmetric bilinear form (M, b) is *primitive* if the quadratic form $q(x) = b(x, x)$ is primitive; if $2 \in R^\times$ then every regular symmetric bilinear form is primitive.

Fact 1.7. *Every regular bilinear form (M, b) over R has a decomposition of the form*

$$(M, q) \cong \begin{cases} <c_1> \perp \ldots \perp <c_r> & \text{if } (M, b) \text{ is primitive} \\ \begin{pmatrix} a_1 & 1 \\ 1 & b_1 \end{pmatrix} \perp \ldots \perp \begin{pmatrix} a_r & 1 \\ 1 & b_r \end{pmatrix} & \text{if } (M, b) \text{ is not primitive} \end{cases}$$

for elements $c_i \in R^\times$ and $a_i, b_i \in \mathfrak{m}$. If $2 \in R^\times$, then any regular quadratic form (M, q) over R has an orthonormal basis

$$(M, q) \cong <a_1> \perp \ldots \perp <a_r>$$

for $a_i \in R^\times$.

Proof. This follows directly from the corresponding statement over the residue field k lifted to R via Fact 1.3. □

2. Witt cancellation and decomposition

Fact 2.1 (Baeza [1, I Cor. 4.3])**.** *A local ring R has the Witt cancellation property: for regular quadratic forms (M_1, q_1), (M_2, q_2), and (M, q), we have $(M_1, q_1) \perp (M, q) \cong (M_2, q_2) \perp (M, q)$ implies $(M_1, q_1) \cong (M_2, q_2)$.*

Remark 2.2. A local ring R (with $2 \notin R^\times$) does not necessarily have the Witt cancellation property for regular bilinear forms.

Fact 2.3. *Every metabolic quadratic form over a local ring R is isomorphic to $H^{\perp r}$, where $H = [0,0]$ is the hyperbolic quadratic plane. In particular, if $2 \in R^\times$, then the same holds for metabolic symmetric bilinear forms. In general, if $2 \notin R^\times$, then there may be many kinds of isomorphism classes of metabolic symmetric bilinear forms, for example $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$, for any $a \in R$.*

Fact 2.4 (Baeza [1, I, Thm. 3.5, III Cor. 4.3], Knebusch [5, §3, Prop. 3])**.** *A local ring R has:*

- *the Witt decomposition property for regular quadratic forms: any regular quadratic form (M, q) has a decomposition $(M, q) \cong (M_0, q_0) \perp H^{\perp n}$, where (M_0, q_0) is anisotropic whose isometry class is uniquely determined by that of (M, q),*
- *(if $2 \in R^\times$) the Witt decomposition property for regular symmetric bilinear forms,*
- *(if $2 \notin R^\times$) the weak Witt decomposition property for regular symmetric bilinear forms: any regular symmetric bilinear form (M, b) has a decomposition $(M, b) \cong (M_0, b_0) \perp N$, where (M_0, q_0) is anisotropic and N is metabolic, neither of whose isometry class is necessarily uniquely determined by that of (M, b).*

Fact 2.5 (O'Meara [8, IX 93:14])**.** *A complete DVR R has the hyperbolic cancellation property: for nondegenerate quadratic forms (M_1, q_1) and (M_2, q_2), if $H \perp (M_1, q_1) \cong H(M_2, q_2)$, then $(M_1, q_1) \cong (M_2, q_2)$.*

3. (Grothendieck–)Witt rings

Recall the Grothendieck–Witt ring $GW(R)$ (resp. $GW_q(R)$) of isometry classes of regular symmetric bilinear (resp. quadratic) forms and the Witt ring $W(R)$ (resp. $W_q(R)$) of isometry classes of regular symmetric bilinear (resp. quadratic) forms modulo hyperbolic forms. Every metabolic bilinear form over an affine scheme is split and is stably hyperbolic, hence $GW(R)$ (resp. $GW_q(R)$) is isomorphic to the Grothendieck ring of isometry classes of regular symmetric bilinear (resp. regular quadratic) forms with respect to orthogonal sum and tensor product. The abelian group $GW_q(R)$ is, via tensor product, a $GW(R)$-algebra. Note that $GW(R)$ and $W(R)$ are rings with unit $<1>$, while if $2 \notin R^\times$ then $GW_q(R)$ and $W_q(R)$ are rings without unit. If $2 \in R^\times$, then we can identify $GW(R)$ and $GW_q(R)$ as well as $W(R)$ and $W_q(R)$.

Fact 3.1. *Let R be a ring with the Witt decomposition property for regular quadratic (resp. symmetric bilinear) forms. Assume furthermore that every metabolic form is hyperbolic. Then classifying isometry classes of quadratic (resp. symmetric bilinear) forms is equivalent to computing the associated (Grothendieck–)Witt group.*

In particular, this holds for regular quadratic forms over a local ring R and for regular symmetric bilinear forms over a local ring with $2 \in R^\times$.

Definition 3.2. Let R be a DVR and π a uniformizer. There exists a unique homomorphism $\partial_\pi : W(K) \to W(k)$ satisfying

$$\partial_\pi(<u\pi^n>) = \begin{cases} <\overline{u}> & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Fact 3.3. *Let R be a DVR and π a uniformizer. Assume $2 \in R^\times$. Then purity holds for the Witt group of symmetric bilinear forms, i.e. the natural sequence*

$$0 \to W(R) \to W(K) \xrightarrow{\partial_\pi} W(k) \to 0$$

is exact.

This raises a general open problem:

Fact 3.4 (Ojanguren–Panin [7])**.** *Let R be a regular local ring containing a field of characteristic $\neq 2$. Then $W(R) \to W(K)$ is injective.*

We have one more result due to Knebusch:

Fact 3.5. *Let R be a DVR. Then $W_q(R) \to W_q(K)$ and $GW_q(R) \to GW_q(K)$ are injective.*

Corollary 3.6. *Let R be a DVR with $2 \neq 0$. Then the natural maps (taking polar bilinear forms) $W_q(R) \to W(R)$ and $GW_q(R) \to GW(R)$ are injective.*

Finally, for a few "non-stable" results (i.e. results about quadratic forms, not (Grothendieck–)Witt classes).

Fact 3.7 (Baeza [1, V Lemma 1.4])**.** *Let R be a complete local ring. If (M, q) is a regular quadratic form such that $(M, q) \otimes_R k$ is isotropic then (M, q) is isotropic. Moreover, the canonical reduction maps $GW_q(R) \to GW_q(k)$ and $W_q(R) \to W_q(k)$ are isomorphisms.*

Fact 3.8 (Parimala–Sridharan [11, Lemma 1.1])**.** *Let R be a complete local ring with 2 invertible. If (M, q) is a regular quadratic form such that $(M, q) \otimes_R k$ is isotropic then (M, q) has a hyperbolic summand.*

Fact 3.9 (Panin [9], Panin–Pimenov [10])**.** *Let R be a (semi)local regular integral domain containing a field such that all residue fields of R are infinite of characteristic $\neq 2$. If $(M, q) \otimes_R K$ is isotropic then (M, q) is isotropic.*

4. Language of lattices

We often study a quadratic (or symmetric bilinear) form over R by going to the generic point, i.e. tensoring by the quotient field K. The language of lattices takes the opposite perspective. Fix a (non-degenerate) quadratic (or symmetric bilinear) form (V, b) over K and consider *lattices* inside V, i.e. projective R-modules $M \subset V$ such that $M \otimes_R K = V$ ("M spans V"). Restricting the form b to a lattice often yields a "form over R". The problem is that, in general, $b|_M$ has values in some arbitrary R-submodule of K, not necessarily in R. The value R-module $\mathfrak{a} = b(M) \subset K$ is often called the *scale* of the lattice. It is a *fractional ideal* of R, i.e. a rank 1 projective R-module $\mathfrak{a} \subset K$.

Since here we do not use the language of lattices, we will instead speak of $\mathfrak{a}$-*valued quadratic forms* over R. Such a form consists of a triple $(M, q, \mathfrak{a})$, where M is a projective R-module, $\mathfrak{a}$ is a projective R-module of rank 1 (which we will often think of as a fractional ideal, i.e. together with an inclusion $\mathfrak{a} \subset K$), and $q : M \to \mathfrak{a}$ is a map satisfying the usual axioms. We also speak of $\mathfrak{a}$-*valued symmetric bilinear forms* $(M, b, \mathfrak{a})$ over R. We will always assume that $\mathfrak{a}$ is actually the R-submodule generated by the values of the form (i.e. the form has scale $\mathfrak{a}$).

Given an R-lattice M inside a symmetric bilinear form (V, b) over K, as well as a fractional ideal $\mathfrak{a} \subset K$, define the $\mathfrak{a}$-*dual* lattice

$$M^{\#\mathfrak{a}} = \{x \in V \,:\, b(x, M) \in \mathfrak{a}\}.$$

Definition 4.1. A lattice M inside a nondegenerate symmetric bilinear form (V, b) over K is called *$\mathfrak{a}$-modular* if $M = M^{\#\mathfrak{a}}$. An R-modular lattice (i.e. satisfying $M = M^{\#}$) is called *unimodular*.

The point is: "unimodular" as a lattice means "regular" as a bilinear form over R. To read more about modular lattices, see O'Meara [8, VIII §82G]. N.B. O'Meara uses "regular lattice" to mean what we call here "lattice inside a nondegenerate bilinear form." For a study of (uni)modular lattices over Dedekind domains, see Fröhlich [4], Bushnell [2], [3], and Wagner [12] (in characteristic 2).

Fact 4.2. *Let (V, b) be a nondegenerate bilinear form over K. There's an equivalence between the category of unimodular lattices inside (V, b) and the category of regular symmetric bilinear forms (M, b) over R together with an K-module isomorphism $M \otimes_R K \to V$.*

But what does $\mathfrak{a}$-modularity mean for a general $\mathfrak{a}$? There is a canonical R-module homomorphism $M^{\#\mathfrak{a}} \to \mathrm{Hom}_R(M, \mathfrak{a})$ given by $x \mapsto y \mapsto b(x, y)$. If (V, b) is nondegenerate over K, then this is actually an R-module isomorphism.

An $\mathfrak{a}$-valued symmetric bilinear form is called *regular* if the natural R-module homomorphism $M \to \mathrm{Hom}_R(M, \mathfrak{a})$ is an isomorphism. An $\mathfrak{a}$-valued quadratic form is called regular if its associated $\mathfrak{a}$-valued symmetric polar bilinear form is. In view of the above statements, we have an analogous statement.

Fact 4.3. *Let (V, b) be a nondegenerate bilinear form over K. There's an equivalence between the category of $\mathfrak{a}$-modular lattices inside (V, b) and the category of regular $\mathfrak{a}$-valued symmetric bilinear forms (M, b) over R together with an K-module isomorphism $M \otimes_R K \to V$.*

Another important way that regular $\mathfrak{a}$-valued forms behave like regular bilinear forms (cf. Fact 1.2).

Fact 4.4. *Let (M, q, a) be an $\mathfrak{a}$-valued quadratic (resp. symmetric bilinear) form over R and $N \subset M$ an R-submodule. Suppose that $\mathfrak{a}$ is the entire module of values. If $(N, q|_N, \mathfrak{a})$ is a regular $\mathfrak{a}$-valued quadratic (resp. symmetric bilinear) form, then there's a decomposition $(M, q, \mathfrak{a}) \cong (N, q|_N, \mathfrak{a}) \perp (N^{\perp}, q|_{N^{\perp}}, \mathfrak{a}')$, where $\mathfrak{a}'$ is the module of values of $q|_{N^{\perp}}$.*

5. Jordan splittings

Definition 5.1. A *Jordan splitting* of an $\mathfrak{a}$-valued symmetric bilinear form $(M, b, \mathfrak{a})$ is a decomposition:

$$(M, b, \mathfrak{a}) = (M_0, b_0, \mathfrak{a}_0) \perp \ldots \perp (M_r, b_r, \mathfrak{a}_r)$$

where the value modules establish a filtration

$$\mathfrak{a} = \mathfrak{a}_0 \supsetneqq \cdots \supsetneqq \mathfrak{a}_r$$

and such that $(M_i, b_i, \mathfrak{a}_i)$ is a regular $\mathfrak{a}_i$-valued symmetric bilinear form for each $0 \leq i \leq r$.

Fact 5.2 (O'Meara [8, IX Thm. 91:9])**.** *The numbers r, $\mathrm{rank}_R(M_0), \ldots, \mathrm{rank}_R(M_r)$ and the value modules $\mathfrak{a}_0 \supsetneqq \cdots \supsetneqq \mathfrak{a}_r$ are invariants of any Jordan splitting of a nondegenerate $\mathfrak{a}$-valued symmetric bilinear form.*

Fact 5.3 (O'Meara [8, IX §91C])**.** *Every nondegenerate $\mathfrak{a}$-valued symmetric bilinear form over a DVR has a Jordan splitting.*

6. Cohomological invariants

Discriminant. A symmetric bilinear form (M,b) of rank n has an associated *(signed) discriminant form*, it's the natural form $\bigwedge^n M \otimes \bigwedge^n M \to R$ given by

$$v_1 \wedge \cdots \wedge v_n \otimes w_1 \wedge \cdots \wedge w_n \mapsto (-1)^{n(n-1)/2} \det(b(v_i,w_j))_{i,j}.$$

If we can choose an R-module isomorphism $\bigwedge^n M \cong R$, then we can view the discriminant form as some rank 1 form $<d>: R\otimes R \to R$, and the class $d = d(b) \in R/R^{\times 2}$ is well-defined. We shall also refer to this class as the *discriminant*. Over a local ring, every rank 1 module is free, so we can always do this.

A symmetric bilinear form (M,b) is unimodular if and only if it's discriminant is a unit, i.e. $d(b) \in R^\times/R^{\times 2}$.

An $\mathfrak{a}$-valued symmetric bilinear form $(M,b,\mathfrak{a})$ of even rank $n = 2m$ also has a signed discriminant form, it's the natural form

$$\left(\bigwedge^n M \otimes \mathfrak{a}^{\vee\otimes m}\right) \otimes \left(\bigwedge^n M \otimes \mathfrak{a}^{\vee\otimes m}\right) \to R$$

given by the similar equation

$$\begin{aligned}(v_1 \wedge \cdots \wedge v_n \otimes f_1 \otimes \cdots \otimes f_m) &\otimes (w_1 \wedge \cdots \wedge w_n \otimes g_1 \otimes \cdots \otimes g_m) \\ &\mapsto (-1)^{n(n-1)/2} f_1 \otimes \cdots \otimes f_m \otimes g_1 \otimes \cdots \otimes g_m \big(\det(b(v_i,w_j))_{i,j}\big),\end{aligned}$$

considering $\det(b(v_i,w_j))_{i,j} \in \mathfrak{a}^{\otimes n}$. The same recipe gives a class $d \in R/R^{\times 2}$.

An $\mathfrak{a}$-valued symmetric bilinear form $(M,b,\mathfrak{a})$ is $\mathfrak{a}$-modular if and only if it's discriminant is a unit, i.e. $d(b) \in R^\times/R^{\times 2}$.

Fact 6.1. *Let R be a local ring. Then there's a canonical isomorphism $R^\times/R^{\times 2} \cong H^1_{\mathrm{fppf}}(R,\mu_2)$.*

Proof. The Kummer sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbb{G}_\mathrm{m} \xrightarrow{2} \mathbb{G}_\mathrm{m} \to 1$$

is an exact sequence of sheaves of groups in the fppf-topology on $\operatorname{Spec} R$. Since R is local, we have $H_{\mathrm{fppf}}(\operatorname{Spec} R, \mathbb{G}_\mathrm{m}) \cong \operatorname{Pic}(R) = 0$. The long exact sequence in cohomology applied to the Kummer sequence then gives the result. □

Therefore, given a regular symmetric bilinear form (M,b) of any rank or a regular $\mathfrak{a}$-valued symmetric bilinear form $(M,b,\mathfrak{a})$, the discriminant gives a class $d(b) \in H^1_{\mathrm{fppf}}(R,\mu_2)$.

Arf invariant. A quadratic form (M,q) of rank n over R has an associated Clifford algebra $C(q) = T(M)/\langle v\otimes v - q(v) : v \in M\rangle$. If q is regular then

$$\begin{cases} C(q) & \text{if } n \text{ even} \\ C_0(q) & \text{if } n \text{ odd}\end{cases}$$

is an Azumaya R-algebra, while

$$\begin{cases} C_0(q) & \text{if } n \text{ even} \\ C(q) & \text{if } n \text{ odd}\end{cases}$$

is an Azumaya algebra over its center, which is an étale quadratic extension S/R.

Fact 6.2. *Let R be a local ring. Then there's a canonical bijection between the set of R-isomorphism classes of étale quadratic R-algebras and $H^1_{\text{ét}}(R,\mathbb{Z}/2\mathbb{Z})$. Also, the canonical map $H^1_{\text{ét}}(R,\mathbb{Z}/2\mathbb{Z}) \to H^1_{\mathrm{fppf}}(R,\mathbb{Z}/2\mathbb{Z})$ is an isomorphism.*

The R-isomorphism class of S/R gives an element of $a(q) \in H^1_{\text{fppf}}(R, \mathbb{Z}/2\mathbb{Z})$ called the *Arf invariant*.

An $\mathfrak{a}$-valued quadratic form $(M, q, \mathfrak{a})$ of rank n over R, while it does not have a Clifford algebra, has an even Clifford algebra $C_0(q) = T(M \otimes M \otimes \mathfrak{a}^\vee)/(I_1 + I_2)$ where we have ideals

$$I_1 = \langle v \otimes v \otimes f - f(q(v)) \,:\, v \in M, f \in \mathfrak{a}^\vee \rangle$$

and

$$I_2 = \langle u \otimes v \otimes f \otimes v \otimes w \otimes g - u \otimes w \otimes f(q(v))g \,:\, u, v, w \in M, f, g \in \mathfrak{a}^\vee \rangle.$$

If $(M, q, \mathfrak{a})$ is a regular $\mathfrak{a}$-valued quadratic form of even rank, then $C_0(q)$ is an Azumaya algebra over its center, which is an étale quadratic extension S/R, the class of which defines the *Arf invariant* $a(q) \in H^1_{\text{fppf}}(R, \mathbb{Z}/2\mathbb{Z})$ for such forms.

So given a regular quadratic form (M, q) of any rank or a regular $\mathfrak{a}$-valued quadratic form $(M, q, \mathfrak{a})$ of even rank, we have two classes: the discriminant of the polar bilinear form $d(b_q) \in H^1_{\text{fppf}}(R, \mu_2)$ and the Arf invariant $a(q) \in H^1_{\text{fppf}}(X, \mathbb{Z}/2\mathbb{Z})$.

Fact 6.3. *The natural homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mu_2$ of sheaves of abelian groups on the* fppf *topology on R gives rise to a group homomorphism $H^1_{\text{fppf}}(X, \mathbb{Z}/2\mathbb{Z}) \to H^1_{\text{fppf}}(X, \mu_2)$ taking $a(q)$ to $d(b_q)$.*

Clifford invariant. If (M, q) is a regular quadratic form over R then the class $c(q) \in {}_2\text{Br}(R)$ of $C(q)$ (or $C_0(q)$ depending on the parity of the rank), is called the *Clifford invariant*.

If $(M, q, \mathfrak{a})$ is a regular $\mathfrak{a}$-valued quadratic form of odd rank then the class $c(q) \in {}_2\text{Br}(R)$ of $C_0(q)$ defines a Clifford invariant.

If $(M, q, \mathfrak{a})$ is a regular $\mathfrak{a}$-valued quadratic form of even rank, there is no canonical way to define a Clifford invariant in this way. What exists is an invariant $\tilde{c}(q) \in {}_2\text{Br}(S)$ which is the Brauer class of $C_0(q)$ over its center S/R. There is also a secondary invariant, defined for regular $\mathfrak{a}$-valued quadratic forms $(M, q, \mathfrak{a})$ of even rank such that $a(q) = 0$. In this case $S \cong R \times R$, which induces a decomposition $C_0(q) \cong C_0^+(q) \times C_0^-(q)$, and the class $c(q) \in {}_2\text{Br}(R)$ of $C_0^+(q)$ is a well-defined invariant of the isometry class of $(M, q, \mathfrak{a})$.

Fact 6.4. *Let R be a local ring. Then $H^2_{\text{fppf}}(R, \mu_2) \cong {}_2\text{Br}(R)$.*

We can thus consider the Clifford invariant (when it's defined), as an element $c(q) \in H^2_{\text{fppf}}(X, \mu_2)$. The Clifford invariant (when it's defined) of a bilinear form is taken to be the Clifford invariant of the quadratic form $b(x, x)$.

Fact 6.5. *Let R be a henselian local ring. Then the canonical reduction map $\text{Br}(R) \to \text{Br}(k)$ is an isomorphism. In particular, if R is a DVR with finite residue field, then $\text{Br}(R) = 0$.*

7. Classification of forms over a DVR

Fact 7.1. *Regular $\mathfrak{a}$-valued symmetric bilinear forms over a p-adic ring with $2 \in R^\times$ are classified up to isometry by their rank and discriminant.*

As a consequence, when $2 \in R^\times$, we can classify nondegenerate symmetric bilinear forms over p-adic rings by means of Jordan splittings.

Fact 7.2 (O'Meara [8, IX Thm. 92:2])**.** *Let R be a p-adic ring. Let a nondegenerate $\mathfrak{a}$-valued symmetric bilinear form $(M, b, \mathfrak{a})$ have Jordan splitting $(M_0, b_0, \mathfrak{a}_0) \perp \ldots \perp$*

$(M_r, b_r, \mathfrak{a}_r)$. *Then the isometry class of* b *is uniquely determined by the list of ranks* $\mathrm{rank}_R(M_0)$, *...*, $\mathrm{rank}_R(M_r)$, *and discriminants* $d(b_0)$, *...*, $d(b_r)$.

When $2 \notin R^\times$, the classification of nondegenerate symmetric bilinear forms due to O'Meara [8, IX Thm. 93:28] does not seem to be of cohomological invariant type! I'm sure the classification of nondegenerate $\mathfrak{a}$-valued quadratic forms is much more clean.

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