Fröhlich twisting via orthogonal motives

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Hopf Algebras and Galois Module Theory

Trace forms

$K/\mathbb{Q}$ finite extension

The *trace form* $q_{L/k} : K \to \mathbb{Q}$

$$q_{K/\mathbb{Q}}(x) = \mathrm{Tr}_{K/\mathbb{Q}}(x^2)$$

$(K, q_{K/\mathbb{Q}})$ nondegenerate quadratic form over $\mathbb{Q}$

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Classical question. What about $K/\mathbb{Q}$ is recovered from $q_{K/\mathbb{Q}}$?

Classical invariants of quadratic forms

k field of characteristic $\neq 2$
(V, q) nondegenerate quadratic form over k

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$b_q : V \times V \to k$ associated bilinear form

$$b_q(v, w) = \frac{1}{2}\big(q(v + w) - q(v) - q(w)\big)$$

$T(q) = \big(b_q(e_i, e_j)\big)$ Gram matrix of q for basis $\{e_1, \ldots, e_n\}$ of V

Definition. The *discriminant* $\operatorname{disc}(q) = \det(T(q)) \in k^\times/k^{\times 2}$

Classical invariants of quadratic forms

$\nu : k \hookrightarrow R$ embedding into a real closed field

$$q \otimes_\nu R \cong < \underbrace{1, \ldots, 1}_{r}, \underbrace{-1, \ldots, -1}_{s} >$$

Definition. The *signature* $\operatorname{sgn}_\nu(q) = r - s$

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$q = \langle a_1, \ldots, a_n \rangle$ diagonalization

Definition. The *Hasse–Witt invariant*

$$\mathrm{hw}_2(q) = \sum_{i<j} (a_i, a_j) \in {}_2\mathrm{Br}(k)$$

where (a, b) is the quaternion algebra over k

$$< x, y \,:\, x^2 = a,\ y^2 = b,\ xy = -yx >$$

Classical invariants of quadratic forms

Classical invariants associated to a quadratic form (V, q) over k

- $\dim(q) \in \mathbb{N}$
- $\mathrm{disc}(q) \in k^{\times}/k^{\times 2}$
- $\mathrm{sgn}_{\nu}(q) \in \mathbb{Z}$
- $\mathrm{hw}_2(q) \in {}_2\mathrm{Br}(k)$

Theorem. (Hasse–Minkowski) Let k be a number field. The dimension, discriminant, signatures, and Hasse–Witt invariant give a complete set of invariants for nondegenerate quadratic forms over k.

Classical invariants of number fields

$K/\mathbb{Q}$ number field

Classical invariants:

- $\dim(q_{K/\mathbb{Q}}) = [K : \mathbb{Q}]$
- $\mathrm{disc}(q_{K/\mathbb{Q}}) = [\Delta_{K/\mathbb{Q}}] \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$
  encodes certain ramified primes in $K/\mathbb{Q}$

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- **Theorem.** (Hermite, Taussky 1968) $\mathrm{sgn}(q_{K/\mathbb{Q}}) = r_1$
  the number of real embeddings

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- **Theorem.** (Hermite, Taussky 1968) $\mathrm{sgn}(q_{K/\mathbb{Q}}) = r_1$ the number of real embeddings
- **Theorem.** (Serre 1984)

  $$\mathrm{hw}_2(q_{K/\mathbb{Q}}) = (2, \mathrm{disc}(q_{K/\mathbb{Q}})) + \mathrm{sw}_2(\rho_{K/\mathbb{Q}})$$

  $\rho_{K/\mathbb{Q}} : \Gamma_{\mathbb{Q}} \to S_n$ permutation Galois action on the embeddings $\mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$
  $\mathrm{sw}_2(\rho_{K/\mathbb{Q}})$ the 2nd Stiefel–Whitney class

2nd Stiefel–Whitney class

Central extension of S_n derived from the "pinor extension"

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbf{Pin}_n(\mathbb{R}) & \longrightarrow & \mathbf{O}_n(\mathbb{R}) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \widetilde{S}_n & \longrightarrow & S_n & \longrightarrow & 1 \end{array}$$

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Connecting homomorphism in Galois cohomology

$$\delta : H^1(\Gamma_{\mathbb{Q}}, S_n) \to H^2(\Gamma_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z}) \cong {}_2\mathrm{Br}(\mathbb{Q})$$

$H^1(\Gamma_{\mathbb{Q}}, S_n) = \{\text{ homomorphisms } \rho : \Gamma_{\mathbb{Q}} \to S_n \}/\text{conjugation}$

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Definition. Let $\rho : \Gamma_{\mathbb{Q}} \to S_n$ be a homomorphism. The *2nd Stiefel–Whitney* class is $\mathrm{sw}_2(\rho) = \delta([\rho])$.

2nd Stiefel–Whitney class

$K/\mathbb{Q}$ Galois extension with group G

$$\rho_{K/\mathbb{Q}} : \Gamma_{\mathbb{Q}} \twoheadrightarrow G \hookrightarrow S_n$$

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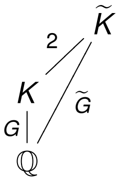
 & & & & & \nwarrow & \uparrow & & \\
 & & & & & & \Gamma_{\mathbb{Q}} & &
\end{array}$$

Inverse Galois theoretic *embedding problem*:

$sw_2(\rho_{K/\mathbb{Q}}) \in {}_2Br(\mathbb{Q})$ vanishes iff the G-Galois extension $K/\mathbb{Q}$ fits into a tower $\widetilde{K}/K/\mathbb{Q}$ with $\widetilde{K}/\mathbb{Q}$ a $\widetilde{G}$-Galois extension?



2nd Stiefel–Whitney class

Example. Let $G = V_4$ Klein four group.

Then $\widetilde{G} = Q_8$ is the quaternion group:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to Q_8 \to V_4 \to 0$$

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(Witt 1936) Let $K = \mathbb{Q}(\sqrt{a}, \sqrt{b})$. Then $K/\mathbb{Q}$ fits into a tower $\widetilde{K}/K/\mathbb{Q}$ with $\widetilde{K}/\mathbb{Q}$ a Q_8-Galois extension if and only if

$$sw_2(\rho_{K/\mathbb{Q}}) = (a, b) + (ab, -1) = 0$$

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Using Serre's formula. Calculate $q_{K/\mathbb{Q}} = \langle 4, 4a, 4b, 4ab \rangle$.

$$\mathrm{hw}_2(q_{K/\mathbb{Q}}) = (a, b) + (a, ab) + (b, ab) = (a, b) + (ab, -1)$$

using bilinearity of the symbol and $(a, a) = (a, -1)$

Fröhlich's perspective

$(\mathbb{Q}^n, q_n)$ sum-of-squares quadratic form

Consider the permutation Galois action $\rho_{K/\mathbb{Q}} : \Gamma_{\mathbb{Q}} \to S_n$ on $\mathbb{Q}^n = \mathrm{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}})$ as an orthogonal representation

$$\rho_{K/\mathbb{Q}} : \Gamma_{\mathbb{Q}} \to \mathbf{O}(q_n)(\overline{\mathbb{Q}})$$

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New Galois action on $\mathbb{Q}^n \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$

$$\sigma(v \otimes \alpha) = \rho_{K/\mathbb{Q}}(\sigma)(v \otimes \sigma(\alpha)), \quad \sigma \in \Gamma_{\mathbb{Q}},\ v \in \mathbb{Q}^n,\ \alpha \in \overline{\mathbb{Q}}$$

Fröhlich's realization. $(K, q_{K/\mathbb{Q}}) = \big((\mathbb{Q}^n \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^{\Gamma_{\mathbb{Q}}}, (q_n \otimes \mathrm{id}_{\overline{\mathbb{Q}}})^{\Gamma_{\mathbb{Q}}}\big)$

Fröhlich's perspective

(V, q) nondegenerate quadratic form

$\rho : \Gamma_{\mathbb{Q}} \to \mathbf{O}(V, q)(\overline{\mathbb{Q}})$ orthogonal representation

New Galois action on $V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$

$$\sigma(v \otimes \alpha) = \rho(\sigma)(v \otimes \sigma(\alpha)), \quad \sigma \in \Gamma_{\mathbb{Q}},\ v \in V,\ \alpha \in \overline{\mathbb{Q}}$$

Fröhlich twist. $(V_\rho, q_\rho) := \left((V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^{\Gamma_{\mathbb{Q}}}, (q \otimes \mathrm{id}_{\overline{\mathbb{Q}}})^{\Gamma_{\mathbb{Q}}}\right)$

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Theorem. (Fröhlich 1985)

$$\mathrm{hw}_2(q_\rho) = \mathrm{hw}_2(q) + (\mathrm{disc}(q), \det(\rho)) + \mathrm{sw}_2(\rho) + sp(\rho)$$

where $sp(\rho)$ is the *spinor class* of ρ

What is Fröhlich twisting?

- Fröhlich 1985
- Jardine 1989 (understand the spinor class)
- Esnault–Kahn–Viehweg 1993 (Serre's formula for finite tame odd covers of Dedekind schemes)
- Kahn 1994 (Serre's formula for arbitrary étale morphisms of schemes)
- Cassou-Noguès–Erez–Taylor 2000 (Serre's formula for finite tame odd covers of arbitrary schemes)
- Cassou-Noguès–Erez–Taylor 2004 (Fröhlich's formula for finite tame odd covers of arbitrary schemes)
- Cassou-Noguès–Chinburg–Morin–Taylor 2011 (Approach to Fröhlich's formula for torsors under finite flat group schemes over affine schemes)
- Saito 2011 (Conjectural Serre's formula for proper even dimensional morphisms of schemes)

Orthogonal Artin motives

Deligne. The category $\mathrm{Mot}_{\mathbb{Q}}^{A}$ of *Artin motives* (i.e. $\mathbb{Q}$-motives of weight 0) is equivalent to the category $\mathrm{Rep}_{\mathbb{Q}}$ of finite dimensional and finite image $\mathbb{Q}$-representations of $\Gamma_{\mathbb{Q}}$.

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Realization functors:

$$H_{\mathrm{B}}, H_{\mathrm{dR}} : \mathrm{Mot}^A_{\mathbb{Q}} \to \mathrm{Vect}_{\mathbb{Q}}, \quad H_\ell : \mathrm{Mot}^A_{\mathbb{Q}} \to \mathrm{Rep}_{\mathbb{Q}_\ell}$$

Definition. $\mathrm{Rep}^O_{\mathbb{Q}}$ category of finite dimensional and finite image orthogonal representations $\rho : \Gamma_{\mathbb{Q}} \to \mathbf{O}(V, q)(\mathbb{Q})$

$\mathrm{QF}_{\mathbb{Q}}$ category of nondegenerate quadratic forms over $\mathbb{Q}$

Orthogonal Artin motives

Theorem (A 2011)

- *There's a commutative diagram of equivalences and forgetful maps:*

$$\begin{array}{ccc} \mathrm{Mot}^{OA}_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{Rep}^{O}_{\mathbb{Q}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Mot}^{A}_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{Rep}_{\mathbb{Q}} \end{array}$$

- *There are isomorphisms of functors, compatible with the above equivalence of categories:*

$$\begin{array}{ccc} \mathrm{Mot}^{OA}_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{Rep}^{O}_{\mathbb{Q}} \\ & {\scriptstyle H_{\mathrm{B}}}\searrow & \downarrow{\scriptstyle f} \\ & & \mathrm{QF}_{\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} \mathrm{Mot}^{OA}_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{Rep}^{O}_{\mathbb{Q}} \\ & {\scriptstyle H_{dR}}\searrow & \downarrow{\scriptstyle Fr\ddot{o}} \\ & & \mathrm{QF}_{\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} \mathrm{Mot}^{OA}_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{Rep}^{O}_{\mathbb{Q}} \\ & {\scriptstyle H_{\ell}}\searrow & \downarrow{\scriptstyle \otimes\mathbb{Q}_\ell} \\ & & \mathrm{Rep}^{O}_{\mathbb{Q}_\ell} \end{array}$$

Orthogonal Artin motives

Under these identifications, Fröhlich's formula is interpreted motivically as a Betti–de Rham–ℓ-adic comparison of invariants:

$$\frac{\mathrm{hw}(H_{\mathrm{dR}}(M))}{\mathrm{hw}(H_{\mathrm{B}}(M))} = \frac{\mathrm{sw}(H_\ell(M))}{sp(H_\ell(M))}$$

This equation takes place in the truncated cohomology ring

$$H^*(\Gamma_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z}) = 1 \oplus H^1(\Gamma_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z}) \oplus H^2(\Gamma_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z})$$

T. Saito's generalization

Serre	Saito
$\mathbb{Q}$ base field	base noetherian $\mathbb{Q}$-scheme Y
finite extension $K/\mathbb{Q}$ of degree n	proper smooth $f : X \to Y$ of even relative dimension d
trace form $q_{K/\mathbb{Q}}$ on K	de Rham cohomology sheaf $H^d_{\mathrm{dR}}(X/Y) = R^d f_* \Omega^\bullet_{X/Y}$ with cup product form $q_{X/Y}$
orthogonal Galois permutation representation $\rho_{K/\mathbb{Q}} : \Gamma_{\mathbb{Q}} \to \mathbf{O}_n(\overline{\mathbb{Q}})$	orthogonal monodromy representation $\rho_{X/Y} : \pi_1(Y, \overline{y}) \to \mathbf{O}(V_{\overline{y}}, q_{\overline{y}})$ associated to the orthogonal ℓ-adic sheaf $(V, q) = (R^d f_* \mathbb{Q}_\ell(\frac{d}{2}), \cup)$ and geometric base point $\overline{y}$ of Y
formula comparing $hw_2(q_{K/\mathbb{Q}})$ and $sw_2(\rho_{K/\mathbb{Q}})$	conjectural formula relating $hw_2(q_{X/Y})$ and $sw_2(\rho_{X/Y})$

T. Saito's generalization

Special case ($Y = \operatorname{Spec}\mathbb{Q}$): Let X be a smooth proper $\mathbb{Q}$-scheme of even dimension d,

$$H^{\bullet}_{\mathrm{dR}}(X/\mathbb{Q}) = \bigoplus_{i=0}^{2d} H^{i}_{\mathrm{dR}}(X/\mathbb{Q})$$

the total de Rham colomology ring,

$$q^{\bullet}_{X/\mathbb{Q}} : H^{\bullet}_{\mathrm{dR}}(X/\mathbb{Q}) \to H^{2d}_{\mathrm{dR}}(X/\mathbb{Q}) \cong \mathbb{Q}$$

the cup product quadratic form,

$$H^{\bullet}_{\ell}(X, \mathbb{Q}_{\ell}) = \bigoplus_{i=0}^{2d} H^{i}_{\ell}(\overline{X}, \mathbb{Q}_{\ell})(\tfrac{d}{2})$$

the total ℓ-adic cohomology ring, and

$$\rho^{\bullet}_{X/\mathbb{Q}} : \Gamma_{\mathbb{Q}} \to \mathbf{O}(H^{\bullet}_{\ell}(X, \mathbb{Q}_{\ell}))$$

the orthogonal Galois monodromy representation.

T. Saito's generalization

Conjecture. (T. Saito 2011)

$$\mathrm{hw}_2(q^\bullet_{X/\mathbb{Q}}) = \mathrm{sw}_2(\rho^\bullet_{X/\mathbb{Q}}) + (2, \mathrm{disc}(q^\bullet_{X/\mathbb{Q}})) + \xi\,\beta_{2,\ell}$$

where

$$\xi = \sum_{j<\frac{d}{2}} (-1)^j \left(\frac{n}{2} - j\right) \chi(X, \Omega^j_{X/\mathbb{Q}})$$

and $\beta_{2,\ell} \in {}_2\mathrm{Br}(\mathbb{Q})$ is ramified at 2 and ℓ

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Theorem. (T. Saito) If $X \subset \mathbb{P}^{d+1}$ smooth hypersurface and $\ell > d+1$, then the conjecture is true.