0.1 Topological Groups

Definition 1. A set G is a *topological group* if G is a group, G is a topological space, and the group operations in G are continuous in the topological space G, i.e. $(a,b) \mapsto ab : G \times G \to G$ and $a \mapsto a^{-1} : G \to G$ are continuous maps.

Definition 2. A topological group is *locally compact* if there exists a compact neighborhood about each point. Examples are $\mathbf{Q}$, $\mathbf{R}^n$, matrix groups...

Definition 3. A topological field K is a field and a topological space such that addition, multiplication, and inversion on the non-zero element are all continuous maps.

I want to classify all locally compact topological fields. I will assume my fields do not have the discrete topology, since any topological group with the discrete topology is locally compact. My goal will be to build a multiplicative homomorphism of the field to the non-negative real number and show that this is in fact, a valuation on the field.

0.2 Haar measure

On any locally compact group G (written additively here) there exists a Haar measure μ, a positive continuous linear functional

$$\mu : C_c(G) \to \mathbf{R}$$

on the space of continuous real function on G with compact support, which is left invariant:

$$\mu(f) = \int_G f(x) d\mu(x) = \int_G f(g+x) d\mu(x), \text{for all } g \in G.$$

This can also be seen in the language of measure theory as a regular σ-additive function on the σ-algebra of Borel $\mathcal{B}$ G, generated by the open sets U of G. I will loosely call vol(U)=μ(U)=$\mu(\chi_U)$ the volume or measure of the set U. Left invariance means $\text{vol}(U) = \text{vol}(g+U)$ for all $g \in G$. We say a measure μ is regular if for all $U \in \mathcal{B}$,

$$\mu(U) = \sup\{\mu(K) : K \subset U, K \text{ is compact}\} = \inf\{\mu(K) : U \subset K, K \text{ is compact}\}.$$

This Haar measure is unique up to a positive constant: Let μ and ν be two Haar measures on a locally compact group G, then there exists a positive constant c such that $\mu = c\nu$.

0.3 Modulus

Let K be a locally compact topological field, and let μ be a Haar measure on the additive group K.

Now let α be an automorphism of K, and define the measure $\alpha(\mu)$ by

$$\alpha(\mu)(U) = \mu(\alpha(U)), \quad \text{for all } U \in \mathcal{B}.$$

Then $\alpha(\mu)$ is invariant and is thus also a Haar measure on K. Thus we have $\alpha(\mu) = m(\alpha) \cdot \mu$, where $m(\alpha)$ only depends on α, and is independent of the choice of Haar measure, since Haar measures are proportional. This is called the *modulus* of the automorphism α. Thus there is a modulus function

$$m : \mathrm{Aut}(K) \to \mathbf{R}_{>0}.$$

Specifically look at automorphisms of of the form $x \mapsto ax$ for any $a \in K^*$. Then I will denote my $m(a)$ the modulus of this automorphism. Thus by definition,

$$\mathrm{vol}(aU) = m(a) \cdot \mathrm{vol}(U), \quad \text{for all } a \in K^*, U \in \mathcal{B}.$$

Now note that for $a, b \in K^*$,

$$m(ab)\mathrm{vol}(U) = \mathrm{vol}(abU) = m(a)\mathrm{vol}(bU) = m(a)m(b)\mathrm{vol}(U),$$

so $m(ab) = m(a)m(b)$, and thus $m : K^* \to R_{>0}$ is a homomorphism. We generally extend the modulus so that $m : K \to R_{\geq 0}$ defining $m(0) = 0$.

Claim 4. There is a positive constant C such that

$$m(x + y) \leq C \max(m(x), m(y)), \quad \text{for all } x, y \in K,$$

where

$$C = \sup_{m(z) \leq 1} m(z + 1)$$

is the smallest such constant.

Proof. Let $x, y \in K$. If $x = y = 0$ then the statement is trivial. Suppose $y \neq 0$ and $m(x) \leq m(y)$. Let $z = xy^{-1}$ thus

$$m(z) = m(x)m(y^{-1}) = m(x)m(y)^{-1} \leq 1.$$

Now we have,

$$m(x+y) = m((xy^{-1}+1)y) = m(z+1)m(y) = C' \max(m(x), m(y)).$$

Where $C' \leq C$ and letting $y = 1$ shows that C is the smallest such constant.

□

By the properties the modulus m has exhibited so far, m is a generalized absolute value.

Definition 5. A generalized absolute value on a field K is a homomorphism $f : K^* \to R_{\geq 0}$ with $f(0) = 0$ defined such that for some $C > 0$,

$$f(x+y) \leq C \max(f(x), f(y)) \quad \text{for all } x, y \in K.$$

If $C = 1$ this is the non-Archimedean absolute value. An absolute value obeying the triangle inequality, $|x+y| \leq |x| + |y| \leq 2\max(|x|, |y|)$, and has $C = 2$, but in fact the converse is also true.

Proof. Notice that

$$f(a_1 + a_2 + a_3 + a_4) \leq C \max(f(a_1+a_2), f(a_3+a_4)) \leq C^2 \max_{1 \leq i \leq 4} f(a_i),$$

or by induction, for $n = 2^r$,

$$f(a_1 + \ldots + a_n) \leq C^r \max f(a_i) = 2^r \max f(a_i) = n \max f(a_i).$$

If n is not a power of two, fill it up with $a_i = 0$ for $n \leq i \leq 2^r$, then

$$f(a_1 + \ldots a_n) \leq 2n \max f(a_i).$$

Now for any $x, y \in K$,

$$\begin{aligned} f((x+y)^n) &\leq f\left(\sum_{i=0..n} \binom{n}{i} x^i y^{n-i}\right) \\ &\leq 2(n+1) \sum f\left(\binom{n}{i}\right) f(x)^i f(y)^{i-1} \\ &\leq 2(n+1) \sum 2 \binom{n}{i} f(x)^i f(y)^{i-1} \\ &= 4(n+1)(f(x)+f(y))^n. \end{aligned}$$

Thus

$$f(x+y) \leq 4^{1/n}(n+1)^{1/n}(f(x)+f(y)) \to f(x)+f(y).$$

□

The square of the regular absolute value has $C = 4$,

$$|x+y|^2 = (|x| + |y|)^2 \leq (2\max(|x|, |y|))^2 = 4\max(|x|^2, |y|^2).$$

Thus any generalized absolute value with $C \leq 2$ makes a field into a metric space. So some power of a generalized absolute value will be a metric on a field. Thus our locally compact field K is a metric space. And the topology generated by m is the same as the topology on K.

Now I will bring in a result from the theory of topological groups.

Claim 6. Let H be a locally compact subgroup of a Housdorff topological group G. Then H is closed.

Applying this to our scenerio, our locally compact field K is closed in its completion (both are metric spaces by the comments above) Thus our locally compact field K is also complete.

Now we can begin to classify by the value of the constant C.

We also need to know:

Claim 7. Any discrete subfield of a non discrete locally compact field of charactoristic 0 is finite.

Proof. This involves first showing that the modulus m is continuous, then showing that closed balls defined by m are compact. Then you have $\{a^n\} \to 0$ in K iff $m(a) < 1$.

Now let F be a discrete subfield. Choose $a \in K$ with $m(a) > 1$, then $m(a^{-n}) = m(a)^{-n} \to 0$, thus $a^{-n} \to 0$ blah blah... □

Case 1: Suppose $C = 1$. Then K has a non-Archimedeann absolute value. Since K has charactoristic 0 and is not discrete by hypothesis, $\mathbf{Q} \subset K$ is not discrete, and so the absolute induces a non-trivial non-Archimedean absolute value onto $\mathbf{Q}$, so by Ostrowski's Theorem this is equivalent to a p-adic absolute value. Since K is complete, it contains the completion of $\mathbf{Q}$ under this absolute value, i.e. $\mathbf{Q}_p \subset K$. Now with some more work one can show that a locally compact normed space over $\mathbf{Q}_p$ is finite dimensional.

Case 2: Suppose $C > 1$. Thus once more, K induces a non-trivial absolute value onto $\mathbf{Q}$, which is not non-Archimedean. So by Ostrowski's Theorem this is equivalent to the regular absolute value on Q and again, since K is complete, K contains $\mathbf{R}$. So K is a real vector space over $\mathbf{R}$, one can show the only two such possibilities are $\mathbf{R}$ and $\mathbf{C}$.