Quadric surface bundles and quaternion algebras

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Hasse Principle

K number field

V_K the set of places of K

K_v completion of K at a place v

q quadratic form over K

$$q = \sum_{i \le j \le n} a_{ij} x_i x_j$$

Theorem (Hasse–Minkowski)

If q has a nontrivial zero over K_v for every place $v \in V_K$ then q has a nontrivial zero over K.

There is a *Hasse Principle* or *Local-Global Principle* for the existence of zeros of quadratic forms over number fields.

Hasse Principle

Question

Over which fields K is there a Hasse Principle for the existence of zeros of quadratic forms?

Results

- $K/\mathbb{F}_p(t)$ global function field (Hasse–Minkowski)
- $K/\mathbb{Q}_p(t)$ function field of a p-adic curve, quadratic forms of rank ≥ 3 (Colliot-Thélène–Parimala–Suresh 2010)

Open Problem

- $K = k(t)$, *k totally imaginary number field, e.g.,* $k = \mathbb{Q}(i)$

Conjecture (Kaplansky 1950s)

Every quadratic form of rank 9 over $\mathbb{Q}(i)(t)$ has a nontrivial zero.

Geometric Surfaces

k algebraically closed field of characteristic $\neq 2$

S smooth projective surface over k

$K = k(S)$

Question

Is there a Hasse Principle for the existence of zeros of quadratic forms over K?

Answer

NO, in general! (Assume $\mathrm{Br}(S)[2] \neq 0$.)

- Brauer class $\alpha \leftrightsquigarrow q_\alpha$ reduced norm of quaternion algebra
- v discrete valuation on K dominates v_D for an irreducible curve $D \subset S$
- completion $K_{v_D} = \mathrm{Frac}(\widehat{\mathscr{O}}_{S,D})$ has residue field $k(D)$
- $\alpha|_{K_v} = 0$ by Tsen's theorem $\iff q_\alpha|_{K_v}$ has a zero
- q_α has no zero over $k(S)$ since α is nontrivial.

Example from Elliptic Curves

Example (Colliot-Thélène–Gabber 2002)

$E_1 : y^2 = (x - a_1)(x - b_1)(x - c_1)$

$E_2 : y^2 = (x - a_2)(x - b_2)(x - c_2)$

nonisogenous elliptic curves over $\mathbb{C}$

$$\mathrm{Br}(E_1 \times E_2) \cong H^1(E_1, \mathbb{Z}/2\mathbb{Z}) \otimes H^1(E_2, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^4$$

generated by the unramified quaternion symbols

$(x - a_1, x - a_2), (x - a_1, x - b_2), (x - b_1, x - a_2), (x - b_1, x - b_2)$

over function field $\mathbb{C}(E_1 \times E_2)$

Rational Surfaces

k algebraically closed field of characteristic $\neq 2$

$\mathrm{Br}(S) = 0$ if S is a rational surface

$K = k(S) = k(t_1, t_2)$

Theorem (A.–Parimala–Suresh 2012)

The Hasse Principle for the existence of zeros of quadratic forms fails over a rational function field K.

Remark

We produce a recipe to construct, and explicitly write down, counterexamples to the Hasse Principle.

General Result

k algebraically closed field of characteristic $\neq 2$
S smooth projective surface over k
Assume (for simplicity) that $\mathrm{Br}(S)[2] = 0$.

Theorem (A.–Parimala–Suresh 2012)

Let $T \to S$ be a flat double cover branched over at most a smooth divisor. Then each nontrivial element $\alpha \in \mathrm{Br}(T)[2]$ gives rise to a quadratic form q_α over $k(S)$ that violates the Hasse Principle for the existence of zeros.

Question

Given S, does there always exist $T \to S$ with $\mathrm{Br}(T)[2] \neq 0$?

Example from Brauer-Manin Obstruction Theory

Example

- $S = \mathbb{P}^2$ and $T \to \mathbb{P}^2$ branched over a smooth sextic curve
- T K3 surface of degree 2 and $\mathrm{Br}(T)[2] \cong (\mathbb{Z}/2\mathbb{Z})^{22-\rho} \neq 0$
- (Hassett–Várilly-Alvarado–Varilly 2011) Construct explicit $\alpha \in \mathrm{Br}(T)[2]$ obstructing weak approximation of $\mathbb{Q}$-points.

$$q_\alpha = \begin{pmatrix} 2(2x+3y+z) & 3x+3y & 3x+4y & 3y^2+2z^2 \\ 3x+3y & 2z & 3z & 4y^2 \\ 3x+4y & 3z & 2(x+3z) & 4x^2+5xy+5y^2 \\ 3y^2+2z^2 & 4y^2 & 4x^2+5xy+5y^2 & 2(2x^3+3x^2z+3xz^2+3z^3) \end{pmatrix}$$

Counterexample to the Hasse Principle over $\mathbb{C}(\frac{x}{z}, \frac{y}{z})$.

Controlled Degeneration

$K = k(S)$ function field of smooth projective surface

Our counterexamples q_α to the Hasse Principle have tightly controlled degeneration.

Definition

A quadratic form q over K has *simple degeneration* if there's a model $q : \mathscr{E} \to \mathscr{L}$ over S that has:

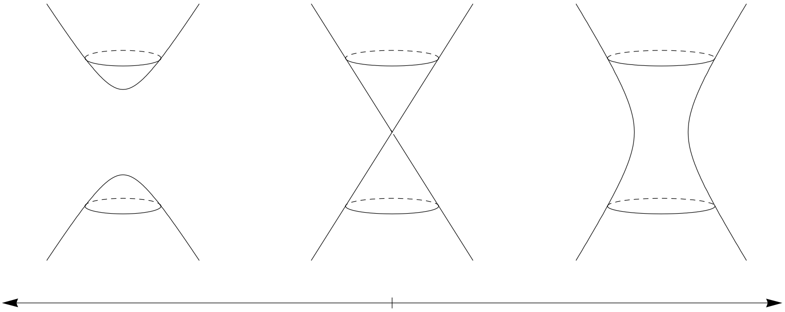
- degenerate fibers with rank 1 radical, along divisor $D \subset S$
- nondegenerate fibers over $S \smallsetminus D$

The geometric counterpart of *square-free discriminant*.

Quadric Fibrations

$q : \mathscr{E} \to \mathscr{L}$ quadratic form with simple degeneration on $D \subset S$

$Q \subset \mathbb{P}_S(\mathscr{E})$ associated quadric fibration over S



$q : \mathscr{E} \to \mathscr{L}$ and $q' : \mathscr{E}' \to \mathscr{L}'$ have quadric fibrations $Q \cong Q'$ $\iff$ q and q' are *projectively similar*, i.e., $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{N}$ and $\mathscr{L}' \cong \mathscr{L} \otimes \mathscr{N}^{\otimes 2}$

A Bijection

S regular integral scheme of dimension ≤ 2 (and $\frac{1}{2} \in \mathscr{O}_S$)

$T \to S$ finite flat, degree 2, with regular branch divisor $D \subset S$

Assume (for simplicity) that $\mathrm{Br}(S)[2] = 0$ and $\mathrm{Pic}(S)[2] = 0$.

Theorem (A.–Parimala–Suresh 2012)

There is a bijection

$$\left\{ \begin{array}{l} Q \to S \textit{ quadric surface fibrations} \\ \textit{with simple degeneration on } D \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \mathscr{A} \textit{ Azumaya quaternion} \\ \textit{algebras on } T \end{array} \right\}$$

induced by the "even Clifford algebra."

Remark

- Hypotheses $\mathrm{Br}(S)[2] = 0$ and $\mathrm{Pic}(S)[2] = 0$ are removable.
- Removing the $\frac{1}{2} \in \mathscr{O}_S$ and/or simple degeneration hypothesis is hard! Leads to "arithmetic statistics" results. (This is joint work with John Voight.)

Even Clifford Algebra

$q : E \to k$ quadratic form over a field k

$$C(q) = T^*(E)/\langle v \otimes v - q(v) \ : \ v \in E \rangle$$

$\mathbb{Z}/2\mathbb{Z}$-graded k-algebra $C(q) = C_0(q) \oplus C_1(q)$

$q : \mathscr{E} \to \mathscr{L}$ quadratic form on a scheme S

(Bichsel 1985) $\mathscr{C}_0(q)$ locally free sheaf of $\mathscr{O}_S$-algebras

Assume $q : \mathscr{E} \to \mathscr{L}$ even rank and simple degeneration

- $\mathscr{Z} \subset \mathscr{C}_0(q)$ center is a flat $\mathscr{O}_S$-algebra of rank 2
- $q : \mathscr{E} \to \mathscr{L}$ has *discriminant cover* $T = \mathbf{Spec}\, \mathscr{Z} \to S$.
- (Kuznetsov) $\mathscr{C}_0(q)$ is Azumaya over T

$$\left\{ \begin{matrix} \text{quadric surface fibrations on } S \\ \text{with simple degeneration on } D \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Azumaya quaternion} \\ \text{algebras on } T \end{matrix} \right\}$$

$$\big(Q \to S\big) \mapsto \big(q : \mathscr{E} \to \mathscr{L}\big) \mapsto \mathscr{C}_0(q) = \mathscr{C}_0(Q)$$

Proof Sketch

$$\left\{\begin{matrix}\text{quadric surface fibrations on } S \\ \text{with simple degeneration on } D\end{matrix}\right\} \to \left\{\begin{matrix}\text{Azumaya quaternion} \\ \text{algebras } \mathscr{A} \text{ on } T\end{matrix}\right\}$$

The hard part is constructing the inverse map.

Ingredients

- Classical "norm form construction" to get $q^U_{\mathscr{A}}$ on $U = S \smallsetminus D$.
- Develop the theory of quadratic forms with simple degeneration over DVRs to get $q^D_{\mathscr{A}}$ over generic point of D.
- "Old-school patching" (in high-tech form) to patch $q^U_{\mathscr{A}}$ and $q^D_{\mathscr{A}}$ to $q_{\mathscr{A}}$ on S retaining simple degeneration.
- For the patching, need to study the orthogonal group scheme $\mathbf{O}(q^D_{\mathscr{A}})$, which is *smooth*, but *not reductive*!
- Isomorphism of group schemes *(kinematics)*

  $$\mathbf{PSO}(q_{\mathscr{A}}) \cong R_{\mathscr{O}[\sqrt{\pi}]/\mathscr{O}}\mathbf{PGL}(\mathscr{A})$$

  degeneration of $D_2 = {}^2A_1$ to $B_1 = A_1$ over a DVR $\mathscr{O}$.

$Y \subset \mathbb{P}^5$ smooth cubic hypersurface over $\mathbb{C}$

Problem

Decide if the generic cubic fourfold is nonrational.

Assume Y contains a plane $P \subset \mathbb{P}^5$.

Projecting from P yields a quadric surface fibration

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, hook] \arrow[dr] & \mathrm{Bl}_P\mathbb{P}^5 \cong \mathbb{P}(\mathscr{O}^3 \oplus \mathscr{O}(-1)) \arrow[d] \\
& \mathbb{P}^2
\end{tikzcd}$$

$\widetilde{Y} \to \mathbb{P}^2$ has simple degeneration along a sextic $D \subset \mathbb{P}^2$
discriminant cover $T \to \mathbb{P}^2$ is a K3 surface of degree 2

Definition

Brauer class of $\mathscr{C}_0(\widetilde{Y})$ is the *Clifford invariant* $\gamma_Y \in \mathrm{Br}(T)[2]$

Lemma (Hassett)

$\gamma_Y = 0 \in \mathrm{Br}(T)[2] \implies$ *Y is rational*

Example from Derived Categories

Example (A.–Bernardara–Bolognesi–Várilly-Alvarado 2012)

Classify minimal rational Y with $\gamma_Y \neq 0$.
Related to Kuznetsov's conjecture on the rationality of cubic fourfolds.

$$q_{\gamma_Y} = \begin{pmatrix} 2(x-4y-z) & -x-3y & x-3y & 2x^2+xz-4y^2+2z^2 \\ & 2(-2y) & x-2y-z & x^2-xy-3y^2+yz-z^2 \\ & & 2x & 2x^2+xy+3xz-3y^2+yz \\ & & & 2(x^3+x^2y+2x^2z-xy^2+xz^2-y^3+yz^2-z^3) \end{pmatrix}$$

These also give counterexamples to the Hasse Principle.

Torelli Theorem for Cubic Fourfolds

$(Y, P) \rightsquigarrow (T, \gamma_Y)$

Corollary (A.–Parimala–Suresh 2012)

If Y containing a plane is generic, then (T, γ_Y) uniquely determines Y up to linear isomorphism.

Implied by Voisin's proof of the Torelli theorem.

Ingredients

- $\mathscr{C}_0(\widetilde{Y})$ defines a γ_Y-twisted sheaf $\mathscr{V}_0$ on T
- $(\mathscr{V}_0, \mathscr{V}_0) = -2 \Longrightarrow M^s_{\gamma_Y}(\mathscr{V}_0)$ at most a point
- Y generic $\Longrightarrow \mathscr{V}_0$ is stable
- $(T, \gamma_Y) \cong (T', \gamma_{Y'}) \Longrightarrow \mathscr{C}_0(\widetilde{Y}) \cong \mathscr{C}_0(\widetilde{Y}')$
- hence $\widetilde{Y} \cong \widetilde{Y}'$ as quadric bundles, so $Y \cong Y'$

Further Thoughts

- An Enriques surface T is a branched double cover of a rational scroll S. Can we use our result to explicate the unique transcendental Brauer class on T? This could be helpful in finding Brauer–Manin obstructions on Enriques surfaces.
- Can we remove the "generic" hypothesis from our results on cubic fourfolds? This could give an "algebraic proof" of the Torelli theorem for cubic fourfolds. This proof could be generalized to other fourfolds birational to quadric surface bundles over surfaces.