1 Distributions on R

Definition 1. Let X be a topological space, let $f : X \to \mathbf{C}$, then define the *support* of f to be the set

$$\operatorname{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}.$$

Definition 2. Let $\Omega\mathbf{R}^n$, then define $\mathcal{D}(\Omega) \subset C^\infty(\Omega)$ to be the space of infinitely differentiable (partials of all orders) functions $f : \mathbf{R}^n \to \mathbf{R}$ which have compact support. We can see that $\mathcal{D}(\Omega)$ is a real vector space, an algebra under pointwise multiplication, and an ideal in $C^\infty(\Omega)$. We can always thing of $\mathcal{D}(\Omega) \subset \mathcal{D}(\mathbf{R}^n)$, and for brevity I will call $\mathcal{D}(\mathbf{R}) = \mathcal{D}$.

Definition 3 (Convergence in $\mathcal{D}(\mathbf{R}^n)$). Let $\{\varphi_i\}$ be a sequence in $\mathcal{D}(\mathbf{R}^n)$ and let $\varphi \in \mathcal{D}(\mathbf{R}^n)$, then we say that $\{\varphi_i\} \to \varphi$ if

1. $\operatorname{supp}(\varphi_i) \subset K$ for some compact $K \subset \mathbf{R}^n$ for all $i \in \mathbf{N}$.
2. $\{D^p\varphi_i\} \to D^p\varphi$ uniformly for each $p \in \mathbf{N}^n$.

This topology defined by some semi-norm?

Theorem 4. The space $\mathcal{D}(\mathbf{R}^n)$ is dense in $C_0(R^n)$, the space of continuous real function of compact support with topology given by uniform convergence, i.e. for any $f \in C_0(R^n)$ with $\operatorname{supp}(f) \subset U$ for some open $U \subset \mathbf{R}^n$ and for any $\varepsilon > 0$, there exists a $\varphi \in \mathcal{D}(\mathbf{R}^n)$ with $\operatorname{supp}(\varphi) \subset U$, and such that for all $x \in \mathbf{R}^n$

$$|f(x) - \varphi(x)| < \varepsilon.$$

Definition 5. A *distribution* T is a continuous linear map $T : \mathcal{D}(\mathbf{R}^n) \to \mathbf{C}$, i.e.

1. $T(\alpha\varphi + \beta\psi) = \alpha T(\varphi) + \beta T(\psi)$ for all $\alpha, \beta \in \mathbf{C}$, $\varphi, \psi \in \mathcal{D}$.
2. $\{\varphi_i\} \to \phi$ in $\mathcal{D} \Rightarrow \{T(\varphi_i)\} \to T(\varphi)$ in $\mathbf{C}$.

The set of all distributions on $\mathbf{R}^n$ will be denoted $\mathcal{D}'(R^n)$. We can see that $\mathcal{D}'(\mathbf{R}^n) \subset \mathcal{D}^*(R^n)$, and assuming the axiom of choice one can show that there exist linear maps $T : \mathcal{D}(\mathbf{R}^n) \to \mathbf{C}$ which are not continuous in the topology of $\mathcal{D}(\mathbf{R}^n)$.

Definition 6. Let $\mathcal{L}(\mathbf{R}^n)$ be the space of all locally integrable functions $f : \mathbf{R}^n \to \mathbf{C}$, i.e. for all compact sets $U \subset \mathbf{R}^n$, f is integrable on U.

Example 7. For every $f \in \mathcal{L}(\mathbf{R}^n)$, we can define a distribution $T_f \in \mathcal{D}'(\mathbf{R}^n)$ given by

$$T_f(\varphi) = \int_{\mathbf{R}^n} f\varphi,$$

for all $\varphi \in \mathcal{D}(\mathbf{R}^n)$, where the integral is the Lebesgue integral, and where really, the integral is finite since the support of φ, hence of $f\varphi$, is bounded.

For any $f \in \mathcal{L}(\mathbf{R}^n)$, the map T_f is obviously linear from the properties of the integral, now let $\{\varphi_i\} \to \varphi$ in $\mathcal{D}$, with $\operatorname{supp}(\varphi_i) \subset K$, then

$$|T_f(\varphi_i) - T_f(\varphi)| \leq \int_{\mathbf{R}^n} |f(\varphi_i - \varphi)| \leq \left(\int_K |f|\right) \sup_{x \in K} |\varphi_i(x) - \varphi(x)| \to 0.$$

So in fact $T_f(\varphi_i) \to T_f(\varphi)$, so T_f is continuous.

Claim 8. Let $f, g \in \mathcal{L}$, then $T_f = T_g \Leftrightarrow f = g$ almost everywhere.

So there is an injection

$$\widetilde{\mathcal{L}} \to \mathcal{D}', \quad f \mapsto T_f,$$

where $\widetilde{\mathcal{L}} = \mathcal{L}/\{f \in \mathcal{L} : f = 0 \text{ almost everywhere}\}$. So we can think of locally integrable functions as distributions. There are distributions which do not accord to locally integrable functions.

Example 9. The *Dirac delta distribution* $\delta \in \mathcal{D}'$ is given by,

$$\delta(\varphi) = \varphi(0), \qquad \text{for all } \varphi \in \mathcal{D}'.$$

Also for each $a \in \mathbf{R}^n$ we can define δ_a in the obvious way. Then obviously δ forms a distribution.

In fact there exists no $f \in \mathcal{L}$ such that $\delta = T_f$, i.e. such that

$$\int_{\mathbf{R}^n} f(x)\varphi(x)dx = \varphi(0), \qquad \text{for all } \varphi \in \mathcal{D}.$$

2 Derivatives of Distributions

We want to develop the notion of the derivative of a distribution, so, we start by looking at derivative of derivatives corresponding to C^1 functions. motivation

Let $f \in C^1(\mathbf{R})$, then for all $\varphi \in \mathcal{D}$,

$$\begin{aligned} T_{f'}(\varphi) &= \int_{\mathbf{R}} f'(x)\varphi(x)dx \\ &= f(x)\varphi(x)\Big|_{-\infty}^{\infty} - \int_{\mathbf{R}} (f(x)\varphi'(x)dx \\ &= -\int_{\mathbf{R}} (f(x)\varphi'(x)dx = -T_f(\varphi'). \end{aligned}$$

This provides the motivation for defining the derivative of an arbitrary distribution $T \in \mathcal{D}'$ by

$$DT(\varphi) = -T(D\varphi), \quad \text{for all } \varphi \in \mathcal{D},$$

and for each $k \in \mathbf{N}$,

$$D^k T(\varphi) = (-1)^k T(D^k \varphi), \quad \text{for all } \varphi \in \mathcal{D}.$$

We can see how this generalizes to $\mathbf{R}^n$, for all $p \in \mathbf{N}^n$,

$$D^p T(\varphi) = (-1)^{|p|} T(D^p \varphi), \quad \text{for all } \varphi \in \mathcal{D},$$

where

$$D^p = \left(\frac{\partial}{\partial x_1}\right)^{p_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{p_n} \quad \text{and} \quad |p| = p_1 + \cdots p_n$$

Note that for any $f \in C^1$, $DT_f = T_{Df}$.

Example 10. Let the *Heavyside function* $H : \mathbf{R} \to \mathbf{R}$ be given by

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}.$$

Then for any $\varphi \in \mathcal{D}$,

$$\begin{aligned} DT_H(\varphi) &= -T_H(\varphi') = -\int_{\mathbf{R}} H(x)\varphi'(x)dx \\ &= -\int_0^{\infty} \varphi'(x)dx = -\varphi(x)\Big|_0^{\infty} = \varphi(0) = \delta(\varphi), \end{aligned}$$

thus we have found out that as distributions $DH = \delta$. Now we want to see what $D\delta$ turns out to be, for any $\varphi \in \mathcal{D}$,

$$D\delta(\varphi) = -\delta(\varphi') = -\varphi'(0),$$

and so in general,

$$D^k\delta(\varphi) = (-1)^k \varphi^{(k)}(0).$$

Example 11. Though the function $f(x) = 1/x$ is not locally integrable, the distribution given by, for every $\varphi \in \mathcal{D}$,

$$T(\varphi) = \mathrm{pv} \int_{\mathbf{R}} \frac{1}{x}\varphi(x)dx = \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \left(\frac{1}{x}\varphi(x)dx \right),$$

does make sense. Let $\varphi \in \mathcal{D}$, and suppose $\mathrm{supp}(\varphi) \subset (-a, a)$, then

3 Multiplication

In general there is no way of multiplying two arbitrary distribution, this is because, even in the case of $\mathcal{L}$, the product of locally integrable function is not necessarily locally integrable, an example is the function $f(x) = 1/\sqrt{|x|}$ which is locally integrable, however, $f^2(x) = 1/|x|$ is not locally integrable. But we can multiply distributions by infinitely differentiable functions. First, for any $f \in \mathcal{L}$, $\alpha \in \mathbf{C}^\infty$, $\varphi \in \mathcal{D}$,

$$T_{\alpha f}(\varphi) = \int_{\mathbf{R}^n} \alpha f \varphi = \int_{\mathbf{R}^n} f(\alpha\varphi) = T_f(\alpha\varphi),$$

since $\mathcal{D}$ is an ideal in $\mathbf{C}^\infty$. Using this as motivation, for any distribution $T \in \mathcal{D}'$, we define

$$(\alpha T)(\varphi) = T(\alpha\varphi), \qquad \text{for all } \varphi \in \mathcal{D}.$$

Example 12. Let $\alpha \in \mathbf{C}^\infty$, then for $\varphi \in \mathcal{D}$,

$$(\alpha\delta)(\varphi) = \delta(\alpha\varphi) = \alpha(0)\varphi(0) = \alpha(0) \cdot \delta(\varphi).$$

In particular

$$\mathrm{id}\delta = 0.$$

Also we have,

$$(\alpha D\delta)(\varphi) = D\delta(\alpha\varphi) = -(\alpha\varphi)'(0) = -\alpha(0)\varphi'(0) - \alpha'(0)\varphi(0) = (\alpha(0)D\delta - \alpha'(0)\delta)(\varphi).$$

In particular

$$\mathrm{id}D\delta = -\delta, \quad \mathrm{id}^2 D\delta = 0, \quad \mathrm{id}D^k\delta = -kD^{k-1}\delta.$$

Theorem 13 (Product rule for distributions). Let $\alpha \in \mathbf{C}^\infty$, $T \in \mathcal{D}'$, then

$$D(\alpha T) = \alpha(DT) + (D\alpha)T.$$

Proof. Let $\varphi \in \mathcal{D}$, then

$$\begin{aligned} D(\alpha T)(\varphi) &= -(\alpha T)(D\varphi) = -T(\alpha D\varphi) \\ &= -T(D(\alpha\varphi) - (D\alpha)\varphi) \\ &= -T(D(\alpha\varphi)) + T((D\alpha)\varphi) \\ &= (\alpha(DT))(\varphi) + (D\alpha)T(\varphi). \end{aligned}$$

□

4 Convergence of Distributions

Definition 14. A sequence of distributions $\{T_i\}$ converges to $T \in \mathcal{D}$ if

$$\{T_i(\varphi)\} \to T(\varphi), \quad \text{for all } \varphi \in \mathcal{D}.$$

This is "pointwise convergence" or "weak" convergence of functionals.

Theorem 15. Let $\{T_i\}$ be a sequence in $\mathcal{D}'$, then if for each $\varphi \in \mathcal{D}$, if $\{T_i(\varphi)\}$ converges in $\mathbf{C}$, then $\{T_i\} \to T$ for some $T \in \mathcal{D}'$.

Theorem 16 (Dominated convergence). Let $\{f_i\}$ be a sequence in $\mathcal{L}$ such that $\{f_i\} \to f$ for some function f such that for all $i \in \mathbf{N}$, $|f_i| \le g$ for some $g \in \mathcal{L}$, then $\{T_{f_i}\} \to T_f$.

Theorem 17. The derivative operator $D : \mathcal{D}' \to \mathcal{D}'$ is linear and continuous, i.e. if $\{T_i\} \to T$ then $\{DT_i\} \to DT$.

Proof. Suppose $\{T_i\} \to T$, then for all $\varphi \in \mathcal{D}$,

$$\{DT_i(\varphi)\} = \{-T_i(D\varphi)\} \to -T(D\varphi) = DT(\varphi).$$

□